Equilibria and Their Stability Do Not Depend on the Control Barrier Function in Safe Optimization-Based Control

Yiting Chen [a], Pol Mestres [b], Jorge Cortes [b], Emiliano Dall'Anese [a]

[a] *Department of Electrical and Computer Engineering, Boston University*

[b] *Department of Mechanical and Aerospace Engineering, University of California, San Diego*

Abstract

Control barrier functions (CBFs) play a critical role in the design of safe optimization-based controllers for control-affine systems. Given a CBF associated with a desired "safe" set, the typical approach consists in embedding CBF-based constraints into the optimization problem defining the control law to enforce forward invariance of the safe set. While this approach effectively guarantees safety for a given CBF, the CBF-based control law can introduce undesirable equilibrium points (i.e., points that are not equilibria of the original system); open questions remain on how the choice of CBF influences the number and locations of undesirable equilibria and, in general, the dynamics of the closed-loop system. This paper investigates how the choice of CBF impacts the dynamics of the closed-loop system and shows that: (i) The CBF does not affect the number, location, and (local) stability properties of the equilibria in the interior of the safe set; (ii) undesirable equilibria only appear on the boundary of the safe set; and, (iii) the number and location of undesirable equilibria for the closed-loop system do not depend of the choice of the CBF. Additionally, for the well-established *safety filters* and controllers based on both *CBF and control Lyapunov functions* (CLFs), we show that the stability properties of equilibria of the closed-loop system are independent of the choice of the CBF and of the associated extended class-$\mathcal{K}$ function.

Key words: Optimization-based controllers, control barrier functions, safety filters, control Lyapunov functions.

1 Introduction

Modern control systems for applications ranging from autonomous driving and robotics, to critical infrastructures such as power grids, require the system to satisfy a set of "safe" operational constraints. Control barrier functions (CBFs) have emerged as a popular and powerful framework to design controllers that ensure forward-invariance of a given set of states termed as safe [4, 24]. CBFs have been used in the constraints of quadratic programs (QPs) associated with the control law in the context of safety filters, where a nominal controller is minimally modified to satisfy the CBF constraint, or in conjunction with control Lyapunov functions (CLFs) in order to guarantee safety and stability. While the ability of CBF-based controllers in ensuring safety is well investigated, there is still a limited understanding of how the choice of CBF influences the behavior of the control-affine system under the CBF-based control law. This paper seeks contributions in this direction by investigating the degree to which the choice of CBF affects the emergence of undesirable equilibria (i.e., spurious equilibria not present in the nominal system and introduced by the CBF-based design) and the local behavior of the closed-loop system.

Literature Review

We rely on the body of work on CBFs [4, 13, 22, 24], which are a well-established tool for rendering a given set forward invariant. A celebrated feature of CBF-based controllers is that they avoid the complex task of computing the system's reachable set, and can be computed efficiently for a variety of control systems. For example, given a nominal controller with desirable stability or optimality properties, CBFs can act on top of it to ensure safety. This technique is often referred to as *safety filters* [7, 9, 21]. If the system is control-affine, the controller can be computed by solving a Quadratic Program (QP) at every point of the state space. CBFs have also been combined with CLFs [18] in order to design controllers with provable forward invariance and asymptotic stability guarantees to the origin. These controllers can also be computed through a QP for control-affine systems and are referred to as CLF-CBF QP controllers. Several works have thoroughly studied the dynamical properties of the closed-loop systems resulting from CBF-based control designs, establishing conditions under which the origin is asymptotically stable [9, 14]; undesirable equilibria emerge [16, 20, 25], and the stability properties of such undesirable equilibria [7].

In general, the study of how the properties of CBF-based controllers depend on the choice of CBF remains largely un-

⋆ This work was supported by the AFOSR Award FA9550-23-1-0740. Corresponding author: Yiting Chen.

Email addresses: yich4684@bu.edu (Yiting Chen), pomestre@ucsd.edu (Pol Mestres), cortes@ucsd.edu (Jorge Cortes), edallane@bu.edu (Emiliano Dall'Anese).

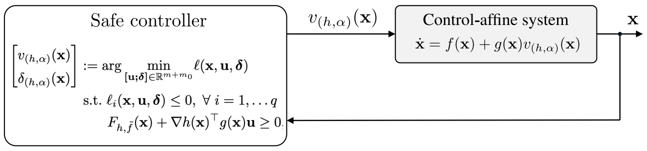

Fig. 1. Closed-loop system that is the subject of the paper. We consider a general formulation for the safe optimization-based controllers that subsumes existing safety filters and CLF-CBF QP approaches.

explored. Indeed, a key open research question is whether the stability properties of the desirable equilibria and the existence and stability properties of undesirable equilibria are dependent or not on the choice of CBF. A notable exception is [2, Lemma IV.3], which shows that the points of discontinuity of CBF-based safety filters are independent of the choice of CBF. This question is of paramount importance for various reasons. If different CBFs induce different undesirable equilibria, one might seek to find the CBF that *minimizes* the number of such undesirable equilibria, for instance. Similarly, if the stability properties of the different equilibria depend on the choice of CBF, one might also seek to find a CBF that makes the desirable equilibria stable (possibly with regions of attraction as large as possible) and the undesirable equilibria unstable.

Statement of Contributions

Our contributions are as follows.

• We consider a control-affine system with a general safe optimization-based controller as shown in Figure 1 (notation will be introduced shortly). The safe controller is "generalized", in the sense that it subsumes existing optimization-based approaches, including the CLF-CBF QP and safety filters. Under a general set of assumptions and for a wide range of CBF-based controllers, we show that:

(i) The number and location of undesirable equilibria of the closed-loop system is independent of the choice of CBF.
(ii) At points where the CBF constraint is not active for a given CBF, the dynamics of the system under the CBF-based controller are independent of the CBF. Therefore, the number, location and stability properties of the desired equilibria are independent of the choice of CBF.
(iii) The trajectories that remain in the boundary of the safe set and the trajectories around the points where the CBF constraint is not active are also independent of the choice of the CBF.

• In Sections 5 and 6, for two widely used CBF-based control designs, namely, the CLF-CBF QP and *safety filters*, we show that also the stability properties of the undesirable equilibria are independent of the choice of CBF. Furthermore, we provide explicit expressions for the Jacobian of the closed-loop system evaluated at undesirable equilibria for both control designs, which show how the dynamics, the CBF and the different parameters used in the control design affect the stability properties of undesirable equilibria.

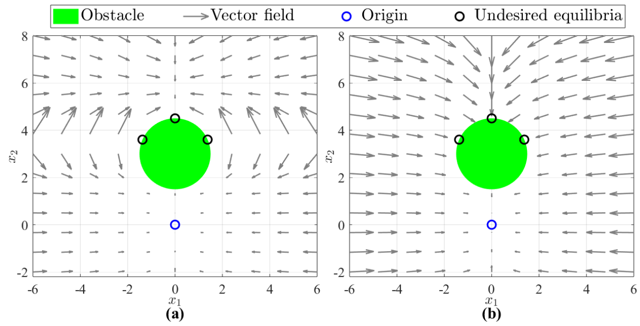

Fig. 2. Examples of undesirable equilibria of an LTI planar system with a CLF-CBF-QP controller for a circular obstacle. The simulation setup and the CLF are as in [16, Fig. 1]. Plots (a) and (b) show the origin (which is a desirable equilibrium), the undesirable (i.e., spurious) equilibria, and the vector field of the closed-loop system for two different choices of the CBF and of the associated extended class-$\mathcal{K}$ function (details are provided in Section 7). The plots show that the number, location, and stability properties of both the desired equilibrium and the undesired equilibria do not change with the CBF pair.

• Finally, we note that some intermediate results in Section 2 are of independent interest, and they characterize the relationship between the gradients and Hessian matrices of two CBFs in the boundary of the safe set.

To gain an intuitive understanding of our results, in Figure 2 we consider the LTI planar system with CLF-CBF-QP controller in [16, Fig. 1]. Figures 2(a) and 2(b) illustrate the undesirable (i.e., spurious) equilibria, and the vector field of the closed-loop system for two different choices of CBF and of the associated extended class-$\mathcal{K}$ function. As it can be observed, even though the two vector fields are significantly different, the number, location and stability properties of the different undesirable equilibria are the same in both cases. Details are provided in Section 7, where additional systems are considered.

2 Preliminaries

Notation. We denote by $\mathbb{Z}_+$, $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ the set of positive integers, real, and nonnegative real numbers, respectively. We write $\text{int}(\mathcal{S}), \partial\mathcal{S}, \bar{\mathcal{S}}$ and $\mathcal{S}^c$ for the interior, the boundary, the closure and the complement of the set $\mathcal{S}$, respectively. Throughout the paper, boldface symbols denote vectors of finite dimension and non boldface symbols denote scalar values. We let $\mathbf{0}_n$ be the n-dimensional zero vector. Given $\mathbf{x} \in \mathbb{R}^n$, $\|\mathbf{x}\|$ denotes its Euclidean norm; for a positive definite matrix $G \in \mathbb{R}^{n \times n}$, we define $\|\mathbf{u}\|_G = \sqrt{\mathbf{u}^\top G \mathbf{u}}$. A function $\beta : \mathbb{R} \to \mathbb{R}$ is of extended class $\mathcal{K}_\infty$ if $\beta(0) = 0$, it is strictly increasing, and $\lim_{t \to \pm\infty} \beta(t) = \pm\infty$. A function $V : \mathbb{R}^n \to \mathbb{R}$ is positive definite if $V(\mathbf{0}_n) = \mathbf{0}_n$ and $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}_n$. Let $a : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $b : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^q$, and consider the optimization problem

$$\underset{\mathbf{u} \in \mathbb{R}^m}{\text{argmin}} \ \ a(\mathbf{x}, \mathbf{u}) \tag{1a}$$
$$\text{s.t. } b(\mathbf{x}, \mathbf{u}) \leq \mathbf{0}_q \tag{1b}$$

for a given $\mathbf{x} \in \mathbb{R}^n$, where the inequality in (1b) is entrywise. Then, given $\mathbf{u} \in \mathbb{R}^m$, we let $\mathcal{I}(\mathbf{x}, \mathbf{u})$ be the set of active constraints of (1) at $(\mathbf{x}, \mathbf{u})$, i.e., $\mathcal{I}(\mathbf{x}, \mathbf{u}) := \{i \in$

$\{1, \dots q\} \ : \ b_i(\mathbf{x}, \mathbf{u}) = 0\}$. Slater's condition holds at $\mathbf{x}$ for Problem (1) if there exists $\hat{\mathbf{u}} \in \mathbb{R}^m$ such that $b_i(\mathbf{x}, \hat{\mathbf{u}}) < 0$ for all $i \in \{1, \dots, q\}$. The Linear Independence Constraint Qualification (LICQ) holds at $(\mathbf{x}, \mathbf{u}) \in \mathbb{R}^n \times \mathbb{R}^m$ for Problem (1) if the vectors $\{\nabla_{\mathbf{u}} b_i(\mathbf{x}, \mathbf{u}), i \in \mathcal{I}(\mathbf{x}, \mathbf{u})\}$ are linearly independent. Given $\mathbf{x} \in \mathbb{R}^n$, a point $(\mathbf{u}_{\mathbf{x}}, \lambda_{\mathbf{x}}) \in \mathbb{R}^m \times \mathbb{R}^q$ is a Karush-Kuhn-Tucker (KKT) point of (1) at $\mathbf{x}$ if it satisfies the following conditions, which we refer to as KKT equations:

$$\nabla_{\mathbf{u}} a(\mathbf{x}, \mathbf{u}_{\mathbf{x}}) + \frac{\partial b}{\partial \mathbf{u}}(\mathbf{x}, \mathbf{u}_{\mathbf{x}})^\top \lambda_{\mathbf{x}} = \mathbf{0}_m, \tag{2a}$$

$$b(\mathbf{x}, \mathbf{u}_{\mathbf{x}}) \leq \mathbf{0}_q, \ \lambda_{\mathbf{x}} \geq \mathbf{0}_q, \tag{2b}$$

$$\lambda_{\mathbf{x}}^\top b(\mathbf{x}, \mathbf{u}_{\mathbf{x}}) = 0. \tag{2c}$$

We refer to (2c) as the *complementary slackness* condition.

2.1 Control Barrier Functions

Consider a control-affine dynamical system of the form

$$\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})\mathbf{u}, \tag{3}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are locally Lipschitz functions, $\mathbf{x} \in \mathbb{R}^n$ is the state, and $\mathbf{u} \in \mathbb{R}^m$ is the input. We let $\mathcal{S} \subset \mathbb{R}^n$ be the safe set.

Definition 2.1 (**Control Barrier Function**): *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that $\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^n : h(\mathbf{x}) \geq 0\}$, $\partial\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^n \ : \ h(\mathbf{x}) = 0\}$ and $\nabla h(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in \partial\mathcal{S}$. The function h is a Control Barrier Function **(CBF)** of the set $\mathcal{S}$ for the system* (3) *if there exists an extended class $\mathcal{K}_\infty$ function α such that for each $\mathbf{x} \in \mathcal{S}$, there exists a control $\mathbf{u} \in \mathbb{R}^m$ satisfying $\nabla h(\mathbf{x})^\top (f(\mathbf{x}) + g(\mathbf{x})\mathbf{u}) + \alpha(h(\mathbf{x})) \geq 0$.* □

Given a safe set $\mathcal{S}$, if $h : \mathbb{R}^n \to \mathbb{R}$ is a CBF of $\mathcal{S}$ and the pair (h, α) satisfies Definition 2.1, then for any $a_1 > 0$, $a_2 > 0$ with $a_2 \geq a_1$, the pair $(a_1 h, a_2 \alpha)$ satisfies Definition 2.1 too. Therefore, if h is a CBF of $\mathcal{S}$, then there are multiple (in fact, infinitely many) pairs satisfying Definition 2.1. Next, given two CBFs of $\mathcal{S}$, the following result presents the relationship between their gradients evaluated at $\partial\mathcal{S}$.

Lemma 2.2 (**Relation between Gradients of CBFs**): *Let $h_1, h_2 : \mathbb{R}^n \to \mathbb{R}$ be two CBFs of $\mathcal{S}$. Then $\nabla h_2(\mathbf{x}) = \zeta(\mathbf{x}) \nabla h_1(\mathbf{x})$ with $\zeta : \partial\mathcal{S} \mapsto \mathbb{R}_{>0}$ a function that is unique.*

PROOF. We follow a similar argument to the one in [2, Lemma IV.3]. Since $\nabla h_1(\mathbf{x}) \neq \mathbf{0}_n$ and $\nabla h_2(\mathbf{x}) \neq \mathbf{0}_n$ for all $\mathbf{x} \in \partial\mathcal{S}$, the sets $\{\mathbf{x} \in \mathbb{R}^n \ : \ h_1(\mathbf{x}) = 0\}$ and $\{\mathbf{x} \in \mathbb{R}^n \ : \ h_2(\mathbf{x}) = 0\}$ define the same differentiable manifold $\partial\mathcal{S}$ of dimension $n-1$ embedded in $\mathbb{R}^n$. By [5, Theorem 3.15], the tangent space of $\partial\mathcal{S}$ at a point $\mathbf{x} \in \partial\mathcal{S}$ is given by $T_{\mathbf{x}} = \ker(\nabla h_1(\mathbf{x})) = \ker(\nabla h_2(\mathbf{x}))$. This implies that $\nabla h_1(\mathbf{x})$ and $\nabla h_2(\mathbf{x})$ are parallel. Moreover, since h_1 and h_2 have the same 0-superlevel set, it follows that $\nabla h_2(\mathbf{x})^\top \nabla h_1(\mathbf{x}) > 0$. This implies that there exists a unique function $\zeta : \partial\mathcal{S} \to \mathbb{R}_{>0}$ satisfying $\nabla h_2(\mathbf{x}) = \zeta(\mathbf{x}) \nabla h_1(\mathbf{x})$ for all $\mathbf{x} \in \partial\mathcal{S}$. □

Given the result in Lemma 2.2, for two CBFs h_1 and h_2 of the set $\mathcal{S}$, we let $\zeta_{(h_1,h_2)} : \partial\mathcal{S} \mapsto \mathbb{R}_{>0}$ be such that $\nabla h_2(\mathbf{x}) = \zeta_{(h_1,h_2)}(\mathbf{x}) \nabla h_1(\mathbf{x})$ for each $\mathbf{x} \in \partial\mathcal{S}$. Next, we study the relationship between the Hessians of two CBFs. To motivate it, we consider the following example.

Example 2.3 (**Relation between Hessians of CBFs**): Consider a ball obstacle with center $\mathbf{x}_c$ and radius r_0. Consider the CBFs $h_1(\mathbf{x}) = \|\mathbf{x} - \mathbf{x}_c\|_2^2 - r_0^2$ and $h_1(\mathbf{x}) = (\|\mathbf{x} - \mathbf{x}_c\|_2^2 - r_0^2) r(\mathbf{x})$, with $r(\mathbf{x}) := \|\mathbf{x}\|_2^2 + 1$. Then, for any $\mathbf{x} \in \partial\mathcal{S}$, $\nabla h_1(\mathbf{x}) = 2(\mathbf{x} - \mathbf{x}_c)$ and $\nabla h_2(\mathbf{x}) = r(\mathbf{x}) \nabla h_1(\mathbf{x}) + h_1(\mathbf{x}) \nabla r(\mathbf{x}) = r(x) \nabla h_1(\mathbf{x})$ due to $h_1(\mathbf{x}) = 0$. Hence, $\zeta_{(h_1,h_2)}(\mathbf{x}) = r(\mathbf{x})$ in this case. If we further evaluate the Hessian matrix at $\mathbf{x} \in \partial\mathcal{S}$, we get that $H_{h_1}(\mathbf{x}) = 2\mathbf{I}_n$ and

$$\begin{aligned} H_{h_2}(\mathbf{x}) &= \nabla h_1(\mathbf{x}) \nabla r(\mathbf{x})^\top + \nabla r(\mathbf{x}) \nabla h_1(\mathbf{x})^\top \\ &\quad + r(\mathbf{x}) H_{h_1}(\mathbf{x}) + h_1(\mathbf{x}) H_r(\mathbf{x}) \\ &= \nabla h_1(\mathbf{x}) \nabla r(\mathbf{x})^\top + \nabla r(\mathbf{x}) \nabla h_1(\mathbf{x})^\top + r(\mathbf{x}) H_{h_1}(\mathbf{x}). \end{aligned}$$

Therefore, the difference between the Hessian of h_1 multiplied by $\zeta_{(h_1,h_2)}(\mathbf{x})$ and the Hessian of h_2 evaluated at $\partial\mathcal{S}$ is equal to the sum of a rank-one matrix and its transpose. □

Inspired by the above example, we define a relation between two CBFs. Formally, given any two pairs (h_1, α_1) and (h_2, α_2) satisfying Definition 2.1, we use the notation $h_1 \overset{\mathrm{H}}{\sim} h_2$ if there exists $\zeta(\mathbf{x}) : \partial\mathcal{S} \mapsto \mathbb{R}_{>0}$ and $\tilde{\zeta}(\mathbf{x}) : \partial\mathcal{S} \mapsto \mathbb{R}^n$ such that for all $\mathbf{x} \in \partial\mathcal{S}$, $\nabla h_2(\mathbf{x}) = \zeta(\mathbf{x}) \nabla h_1(\mathbf{x})$ and

$$H_{h_2}(\mathbf{x}) = \nabla h_1(\mathbf{x}) \tilde{\zeta}(\mathbf{x})^\top + \tilde{\zeta}(\mathbf{x}) \nabla h_1(\mathbf{x})^\top + \zeta(\mathbf{x}) H_{h_1}(\mathbf{x}).$$

We have the following general result.

Proposition 2.4 (**Equivalence Relation**): *$\overset{\mathrm{H}}{\sim}$ is an equivalence relation.*

PROOF. We need to show that "$\overset{\mathrm{H}}{\sim}$" is (a) reflexive: $h \overset{\mathrm{H}}{\sim} h$ for any h; (b) symmetric: $h_2 \overset{\mathrm{H}}{\sim} h_1$ if $h_1 \overset{\mathrm{H}}{\sim} h_2$; and (c) transitive: $h_1 \overset{\mathrm{H}}{\sim} h_3$ if $h_1 \overset{\mathrm{H}}{\sim} h_2$ and $h_2 \overset{\mathrm{H}}{\sim} h_3$.

(a) Reflexivity. Taking $\zeta \equiv 1$ and $\tilde{\zeta} \equiv 0$, it follows that $h \overset{\mathrm{H}}{\sim} h$.

(b) Symmetry. Suppose that $\nabla h_2(\mathbf{x}) = \zeta(\mathbf{x}) \nabla h_1(\mathbf{x})$ and $H_{h_2}(\mathbf{x}) = \nabla h_1(\mathbf{x}) \tilde{\zeta}(\mathbf{x})^\top + \tilde{\zeta}(\mathbf{x}) \nabla h_1(\mathbf{x})^\top + \zeta(\mathbf{x}) H_{h_1}(\mathbf{x})$. It follows that $\nabla h_1(\mathbf{x}) = \frac{1}{\zeta(\mathbf{x})} \nabla h_2(\mathbf{x})$ and

$$H_{h_1} = \frac{1}{\zeta} H_{h_2} - \nabla h_2 \frac{\zeta_{(h_2,h_1)}}{\zeta} \tilde{\zeta}^\top - \frac{\zeta_{(h_2,h_1)}}{\zeta} \tilde{\zeta} \nabla h_2^\top,$$

where $\zeta_{(h_2,h_1)}$ satisfying $\nabla h_1(\mathbf{x}) = \zeta_{(h_2,h_1)}(\mathbf{x}) \nabla h_2(\mathbf{x})$ by Lemma 2.2.

(c) Transitivity. Suppose that

$$\begin{aligned} \nabla h_2(\mathbf{x}) &= \zeta_1(\mathbf{x}) \nabla h_1(\mathbf{x}), \\ H_{h_2}(\mathbf{x}) &= \nabla h_1(\mathbf{x}) \tilde{\zeta}_1(\mathbf{x})^\top + \tilde{\zeta}_1(\mathbf{x}) \nabla h_1(\mathbf{x})^\top \end{aligned}$$

$$
\begin{aligned}
&+ \zeta_1(\mathbf{x}) H_{h_1}(\mathbf{x}),\\
\nabla h_3(\mathbf{x}) =& \zeta_2(\mathbf{x}) \nabla h_2(\mathbf{x}),\\
H_{h_3}(\mathbf{x}) =& \nabla h_2(\mathbf{x}) \tilde{\zeta}_2(\mathbf{x})^\top + \tilde{\zeta}_2(\mathbf{x}) \nabla h_2(\mathbf{x})^\top\\
&+ \zeta_2(\mathbf{x}) H_{h_2}(\mathbf{x}).
\end{aligned}
$$

It follows that $\nabla h_3(\mathbf{x}) = \zeta_2(\mathbf{x})\zeta_1(\mathbf{x})\nabla h_1(\mathbf{x})$ and

$$
\begin{aligned}
H_{h_3} =& \nabla h_1 \zeta_{(h_1,h_2)} \tilde{\zeta_2}^\top + \tilde{\zeta}_2 \zeta_{(h_1,h_2)} \nabla h_1^\top\\
&+ \zeta_2(\nabla h_1 \tilde{\zeta}_1^\top + \tilde{\zeta}_1 \nabla h_1^\top + \zeta_1 H_{h_1})\\
=& \nabla h_1 (\zeta_{(h_1,h_2)} \tilde{\zeta}_2 + \zeta_2 \tilde{\zeta}_1)^\top\\
&+ (\tilde{\zeta}_2 \zeta_{(h_1,h_2)} + \zeta_2 \tilde{\zeta}_1) \nabla h_1^\top + \zeta_2 \zeta_1 H_{h_1}.
\end{aligned}
$$

□

Next, we characterize a class of CBFs which are equivalent in the sense of $\overset{\mathrm{H}}{\sim}$.

Given a pair (h, α), a continuously differentiable extended class $\mathcal{K}$ function $\gamma : \mathbb{R} \to \mathbb{R}$, a continuously differentiable positive function $\eta : \mathbb{R}^n \to \mathbb{R}$, and $a, b \in \mathbb{R}_{\geq 0}$ with $a+b > 0$, we consider the functional operation $\phi(h)$, where $\phi(h) : \mathbb{R}^n \to \mathbb{R}$ is given by $\phi(h)(x) = a\gamma(h)(x) + b\eta(x)h(x)$ for all $x \in \mathbb{R}^n$. Given a continuously differentiable function h, its transform $\phi(h)$ is also a continuously differentiable function with the same domain and the same 0-superlevel set. We should point out that $\phi(h)$ may not be a valid CBF, even if h is. The following result shows that if h and $\phi(h)$ are CBFs, then they are equivalent.

Proposition 2.5 (**Conditions for Equivalence of CBF and its Transform**): *Let $h_1 : \mathbb{R}^n \to \mathbb{R}$, $h_2 : \mathbb{R}^n \to \mathbb{R}$ be CBFs of $\mathcal{S}$. Suppose that there exists $\gamma : \mathbb{R} \to \mathbb{R}$, $\eta : \mathbb{R}^n \to \mathbb{R}$, $a \geq 0$, and $b \geq 0$ such that:*

- *$h_2(\mathbf{x}) = a\gamma(h_1(\mathbf{x})) + b\eta(\mathbf{x})h_1(\mathbf{x})$, for all $\mathbf{x} \in \mathbb{R}^n$;*
- *γ is twice continuously differentiable on an open set containing 0;*
- *η is twice continuously differentiable on an open set containing $\partial\mathcal{S}$;*
- *$a + b > 0$.*

Then $h_1 \overset{H}{\sim} h_2$.

PROOF. By differentiating $h_2(\mathbf{x}) = a\gamma(h_1(\mathbf{x})) + b\eta(\mathbf{x})h_1(\mathbf{x})$ we obtain

$$\nabla h_2(\mathbf{x}) = (a\gamma'(h_1(\mathbf{x})) + b\eta(\mathbf{x}))\nabla h_1(\mathbf{x}) + bh_1(\mathbf{x})\nabla\eta(\mathbf{x})\,.$$

By differentiating one more time we obtain the desired expression by taking $\zeta(\mathbf{x}) = a\gamma'(0) + b\eta(\mathbf{x})$, $\tilde{\zeta}(\mathbf{x}) = \frac{1}{2}a\gamma''(0)\nabla h_1(\mathbf{x}) + b\nabla\eta(\mathbf{x})$. □

By Proposition 2.5, given a CBF h, one can define a class of functions

$$
\begin{aligned}
\Psi^1(h) := \{ a\cdot\gamma(h) + b\cdot\eta\cdot h :\ &\forall\, a, b, \gamma, \eta \text{ satisfying}\\
&\text{assumptions in Proposition 2.5}\},
\end{aligned}
$$

$$
\begin{aligned}
\Psi^i(h) := \{ a\cdot\gamma(\tilde{h}) + b\cdot\eta\cdot\tilde{h} :\ &\forall\, a, b, \gamma, \eta \text{ satisfying}\\
&\text{assumptions in Proposition 2.5}, \forall\, \tilde{h} \in \Psi^{i-1}(h)\},
\end{aligned}
$$

for any positive integer $i \geq 2$. Here, $\Psi^i(h)$ is the set of function obtained from applying i times on h an operation of the type ϕ, where a, b, γ, η can be different in each step of the composition. The following result is a consequence of Propositions 2.4 and 2.5.

Corollary 2.6 (**Equivalence between the Functions in $\bigcup_{i\in\mathbb{Z}_+}\Psi^i(h)$**): *Let h, h_1, and h_2 be CBFs of a safe set $\mathcal{S} \subset \mathbb{R}^n$. Suppose that $h_1, h_2 \in \bigcup_{i\in\mathbb{Z}_+}\Psi^i(h)$. Then $h_1 \overset{H}{\sim} h_2$.* □

Corollary 2.6 asserts that, if two CBFs are synthesized from multi-composite operations based on the same CBF h, then the two CBFs are equivalent.

2.2 Control Lyapunov Functions

We recall the definition of Control Lyapunov Function.

Definition 2.7 (**Control Lyapunov Function**): *Given an open set $\mathcal{D} \subseteq \mathbb{R}^n$ and a point $\mathbf{x}^* \in \mathcal{D}$, a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ is a control Lyapunov function **(CLF)** for system (3) with respect to $\mathbf{x}^*$ if:*

(i) V is proper in $\mathcal{D}$, i.e., $\{\mathbf{x} \in \mathcal{D} : V(\mathbf{x}) \leq c\}$ is a compact set for all $c > 0$,
(ii) $V(\mathbf{x}^*) = 0$ and $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{x}^*$,
(iii) there exists a class-$\mathcal{K}$ function β such that for each $\mathbf{x} \in \mathbb{R}^n$, there exists $\mathbf{u} \in \mathbb{R}^m$ satisfying

$$\nabla V(\mathbf{x})^\top (f(\mathbf{x}) + g(\mathbf{x})\mathbf{u}) + \beta(V(\mathbf{x})) \leq 0. \tag{4}$$

□

Any locally-Lipschitz controller $\mathbf{u} : \mathbb{R}^n \to \mathbb{R}^m$ satisfying (4) for all $\mathbf{x} \in \mathbb{R}^n$ renders the equilibrium $\mathbf{x}^*$ globally asymptotically stable [18].

3 Problem Statement

Consider a control-affine system of the form (3) with f and g continuously differentiable subject to $\mathbf{u} = v(\mathbf{x})$,

$$\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})v(\mathbf{x}), \tag{5}$$

where $v : \mathbb{R}^n \to \mathbb{R}^m$ is defined as the unique solution of the following optimization problem

$$
\begin{aligned}
\begin{bmatrix} v(\mathbf{x}) \\ \delta(\mathbf{x}) \end{bmatrix} = \arg &\min_{[\mathbf{u};\boldsymbol{\delta}]\in\mathbb{R}^{m+m_0}} \ell(\mathbf{x}, \mathbf{u}, \boldsymbol{\delta}) \\
&\text{s.t. } \ell_i(\mathbf{x}, \mathbf{u}, \boldsymbol{\delta}) \leq 0,\ \forall i = 1, \ldots q.
\end{aligned} \tag{6}
$$

Here, $\ell : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m_0} \to \mathbb{R}$ is the objective function and $\ell_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m_0} \to \mathbb{R}$ for $i = 1, \ldots, q$, $q \in \mathbb{Z}_+$ are the constraints (assumptions on objective and constraints will be explained shortly). Moreover, $\mathbf{x}$ is the state and $\boldsymbol{\delta}$ is an auxiliary optimization variable. Assume that $\mathbf{x}^* \in \mathbb{R}^n$ is such that $f(\mathbf{x}^*) + g(\mathbf{x}^*)v(\mathbf{x}^*) = 0$. We build our model on the system (5) and the optimization-based controller (6) – and subsequently add complexity to (6) by adding CBF-based safety constraints – with the intent of capturing several classes of optimization-based controller setups.

Remark 3.1 (**Connection with Optimization-based Controllers in the Literature**): The framework described above encompasses various existing setups considered in the literature. Specifically:

Optimization-based Controller on top of Nominal Controller: Consider the control-affine system $\dot{\mathbf{x}} = \bar{f}(\mathbf{x}) + g(\mathbf{x})\mathbf{u}$ under the nominal controller $\mathbf{u} = k(\mathbf{x})$. Then, the system (5) with $f(\mathbf{x}) = \bar{f}(\mathbf{x}) + g(\mathbf{x})k(\mathbf{x})$ corresponds to the optimization-based controller acting on top of the nominal controller;

Pointwise Minimum-norm Control Optimization: Given a CLF V for system (3) with respect to a certain point $\mathbf{x}^*$, the pointwise minimum-norm controller [12, Chapter 4.2] finds the minimum-norm controller satisfying the CLF condition (4). This can be obtained from (6) by taking $\ell(\mathbf{x}, \mathbf{u}, \boldsymbol{\delta}) = \|\mathbf{u}\|^2$, $q = 1$ and $\ell_1(\mathbf{x}, \mathbf{u}, \delta) = \nabla V(\mathbf{x})^\top (f(\mathbf{x}) + g(\mathbf{x})\mathbf{u}) + \beta(V(\mathbf{x}))$;

Stabilization under Model Uncertainty: If system (3) is unknown but the uncertainty can be modeled as a Gaussian Process, and a CLF V for the true system (3) is available, [6] shows that a controller of the form (6) can be used to robustly stabilize the system;

Input Clipping: Suppose that $k(\mathbf{x})$ is a nominal controller for system (3), and consider the case where $\ell(\mathbf{x}, \mathbf{u}, \boldsymbol{\delta}) = \|\mathbf{u} - k(\mathbf{x})\|_2^2 + \|\boldsymbol{\delta}\|_2^2$ and $\ell_i(\mathbf{x}, \mathbf{u}, \boldsymbol{\delta}) = u_i - u_{i,\text{sat}}$, where $u_{i,\text{sat}}$ is a (saturation) limit for the ith input. Then, one has that $\delta(\mathbf{x}) = 0$ and $v(\mathbf{x}) = \min(k(\mathbf{x}), \mathbf{u}_{\text{sat}})$, where the minimum is taken entry-wise. A similar problem can be written when one has lower and upper limits on the inputs.

Note also that, in general, one can "deactivate" the controller $v(\mathbf{x})$ by simply setting $\ell_i \equiv 0$ and $\ell(\mathbf{x}, \mathbf{u}, \boldsymbol{\delta}) = \|\mathbf{u}\|_2^2 + \|\boldsymbol{\delta}\|_2^2$, yielding $v(\mathbf{x}) \equiv 0$ in (6). □

In general, the closed-loop dynamics $f + gv$ might enjoy desirable properties such as stability, optimality, or robustness, but might also be unsafe. We seek to modify the optimization problem defining $v(\mathbf{x})$ to achieve safety guarantees. Before presenting our control design, let (h, α) be a pair satisfying Definition 2.1. We make the following assumptions.

Assumption 3.2 (**Differentiability and Convexity of Objective Function and Constraints**): *The functions* $\ell : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m_0} \to \mathbb{R}$ *and* $\{\ell_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m_0} \to \mathbb{R}\}_{i=1}^q$ *are such that:*

- *for any fixed* $\mathbf{x} \in \mathbb{R}^n$ *and any* $1 \leq i \leq q$, $\ell_i(\mathbf{x}, \cdot, \cdot)$ *is convex w.r.t* $[\mathbf{u}; \boldsymbol{\delta}]$ *and* $\ell(\mathbf{x}, \cdot, \cdot)$ *is strongly convex w.r.t* $[\mathbf{u}; \boldsymbol{\delta}]$.
- ℓ, $\nabla_{[\mathbf{u};\boldsymbol{\delta}]}\ell$, ℓ_i *and* $\nabla_{[\mathbf{u};\boldsymbol{\delta}]}\ell_i$ *,*$i = 1, ..., q$, *are continuously differentiable w.r.t* $[\mathbf{x}; \mathbf{u}; \boldsymbol{\delta}]$*;*
- $\nabla h(\mathbf{x})$ *and* $\alpha(\cdot)$ *are continuously differentiable.* □

Now, consider the following optimization problem:

$$\begin{bmatrix} v_{(h,\alpha)}(\mathbf{x}) \\ \delta_{(h,\alpha)}(\mathbf{x}) \end{bmatrix} := \arg \min_{[\mathbf{u};\boldsymbol{\delta}] \in \mathbb{R}^{m+m_0}} \ell(\mathbf{x}, \mathbf{u}, \boldsymbol{\delta}) \tag{7}$$
$$\text{s.t. } \ell_i(\mathbf{x}, \mathbf{u}, \boldsymbol{\delta}) \leq 0, \ \forall\, i = 1, \dots q$$
$$F_{h,f}(\mathbf{x}) + \nabla h(\mathbf{x})^\top g(\mathbf{x})\mathbf{u} \geq 0,$$

where

$$F_{h,f}(\mathbf{x}) = \nabla h(\mathbf{x})^\top f(\mathbf{x}) + \alpha(h(\mathbf{x})).$$

Note that (6) is the version of (7) without the CBF constraint, which is introduced to achieve safety guarantees. Next we illustrate different instantiations of (7) often used in the literature.

Example 3.3 (**CLF-CBF QP**): Here we focus on a specific case of (7) that emerges when using CBF constraints in conjunction with CLF constraints (see also Remark 3.1). We let $V : \mathbb{R}^n \to \mathbb{R}$ be a CLF with respect to a certain point $\mathbf{x}^*$. Since the two constraints might not be simultaneously feasible, we include a relaxation variable $\delta \in \mathbb{R}$ that adds a slack term to the CLF constraint. Our goal is to find the minimum-norm control input (weighted by a positive definite matrix-valued function $G : \mathbb{R}^n \to \mathbb{R}^{m \times m}$) that satisfies the CLF and CBF constraints. This leads to the following optimization-based controller:

$$\begin{bmatrix} \bar{v}_{(h,\alpha)}(\mathbf{x}) \\ \bar{\delta}_{(h,\alpha)}(\mathbf{x}) \end{bmatrix} = \arg \min_{[\mathbf{u};\delta] \in \mathbb{R}^{m+1}} \frac{1}{2}\|\boldsymbol{u}\|_{G(\mathbf{x})}^2 + \frac{1}{2}p\delta^2 \tag{8}$$
$$\text{s.t. } F_{V,f}(\mathbf{x}) + \nabla V(\mathbf{x})^\top g(\mathbf{x})\mathbf{u} \leq \delta,$$
$$F_{h,f}(\mathbf{x}) + \nabla h(\mathbf{x})^\top g(\mathbf{x})\mathbf{u} \geq 0,$$

where $F_{h,f}(\mathbf{x}) = \nabla h(\mathbf{x})^\top f(\mathbf{x}) + \alpha(h(\mathbf{x}))$, $F_{V,f}(\mathbf{x}) = \nabla V(\mathbf{x})^\top f(\mathbf{x}) + \beta(V(\mathbf{x}))$, and $p > 0$ is a constant parameter set by the user. We note that the CLF-CBF QP in (8) reduces to: (i) the formulation used in [3] if $k(\mathbf{x}) = 0$ and $G(\mathbf{x}) = \mathbf{I}_m$; (ii) the version considered in [20] if $G(\mathbf{x}) = \mathbf{I}_m$. Note also that (8) reduces to (7) by letting $m_0 = 1$, $q = 1$, $l_1(\mathbf{x}, \mathbf{u}, \boldsymbol{\delta}) = F_{V,f}(\mathbf{x}) + \nabla V(\mathbf{x})^\top g(\mathbf{x})\mathbf{u} - \delta$ and $l(\mathbf{x}, \mathbf{u}, \boldsymbol{\delta}) = \frac{1}{2}\|\boldsymbol{u}\|_{G(\mathbf{x})}^2 + \frac{1}{2}p\delta^2$. □

Example 3.4 (**Safety Filter**): Given a nominal stabilizing controller $k : \mathbb{R}^n \to \mathbb{R}^m$ (which often is assumed to stabilize a certain desirable equilibrium $\mathbf{x}^*$), safety filters seek to find the control input that is closest to k (with a distance induced by a positive definite matrix-valued function $G : \mathbb{R}^n \to \mathbb{R}^{m \times m}$) and satisfies the CBF constraint. This leads to the closed-loop system $\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})k(\mathbf{x}) + g(\mathbf{x})\breve{v}_{(h,\alpha)}(\mathbf{x})$ associated with the following optimization-based control-law:

$$\breve{v}_{(h,\alpha)}(\mathbf{x}) = \arg \min_{\mathbf{u} \in \mathbb{R}^n} \|\mathbf{u}\|_{G(\mathbf{x})}^2 \tag{9}$$
$$\text{s.t. } \nabla h(\mathbf{x})^\top (f(\mathbf{x}) + g(\mathbf{x})(k(\mathbf{x}) + \mathbf{u})) + \alpha(h(\mathbf{x})) \geq 0.$$

We note that (9) is a particular case of (7), obtained by letting $m_0 = 0$, $q = 0$ and $l(\mathbf{x}, \mathbf{u}, \boldsymbol{\delta}) = \frac{1}{2}\|\mathbf{u}\|_{G(\mathbf{x})}^2$. □

Other CBF-based control designs, such as penalty function-based [14], input-constrained [1], and multiple CBF-based controllers [23] can be cast as (7) with an appropriate choice of the functions ℓ and $\{\ell_i\}_{i=1}^q$.

Next, we make the following assumption on (7).

Assumption 3.5 (**Slater's Condition**): *For all $\mathbf{x} \in \mathcal{S}$, problem* (7) *satisfies Slater's condition.* □

Note that Assumption 3.5 implies that problem (6) satisfies Slater's condition as well; moreover, Assumption 3.5 is equivalent to (7) and (6) being strictly feasible (i.e, all inequalities being feasible with a strict inequality). Note further that, for any given $\mathbf{x} \in \mathcal{S}$, the optimization problems (7) and (6) each have a unique solution; this means that $v_{(h,\alpha)}(\mathbf{x})$ and $v(\mathbf{x})$ are well-defined for all $\mathbf{x} \in \mathcal{S}$.

Now, consider the system

$$\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})v_{(h,\alpha)}(\mathbf{x}), \tag{10}$$

which we refer to as the *filtered system* (because the constraint $F_{h,f}(\mathbf{x}) + \nabla h(\mathbf{x})^\top g(\mathbf{x})\mathbf{u} \geq 0$ is included in the computation of the control input). Correspondingly, we refer to (5) as the *unfiltered system*. Given any pair (h, α) satisfying Definition 2.1, the set $\mathcal{S}$ is forward-invariant for the filtered system but may not be forward-invariant for the unfiltered one. Throughout the paper, we refer to an equilibrium of the unfiltered system (5) as a *desirable equilibrium* and an equilibrium of the filtered system (10) that is not an equilibrium of the unfiltered system (5) as an *undesirable equilibrium*.

Our goal is to provide answers to the following questions:

(1) How do the equilibria of (10) depend on the choice of the pair (h, α)?
(2) How do the dynamical properties of (10) depend on the pair (h, α)?

Before proceeding, we note that under Assumptions 3.2 and 3.5, [11, Theorem 5.3] shows that $v_{(h,\alpha)}(\mathbf{x})$ and $v(\mathbf{x})$ are continuous for all $\mathbf{x} \in \mathcal{S}$. Therefore, (5) and (10) have a solution for every initial condition in $\mathcal{S}$. However, since $v_{(h,\alpha)}$ and v are in general only continuous, but not locally Lipschitz, this solution might not be unique. However, under different concrete instantiations of (5) and (10) considered throughout the paper, Assumptions 3.2 and 3.5 are sufficient to guarantee that the corresponding $v_{(h,\alpha)}$ and v are locally Lipschitz. We refer the interested reader to [15] for a survey on regularity properties of such optimization-based controllers.

4 Impact of CBF on Dynamics and Equilibria

In this section, we start examining how the choice of the pair (h, α) affects the dynamics of the filtered system (10). In particular, we show that: (i) the dynamics of the filtered system do not depend on the pair (h, α) at points where the CBF condition is satisfied strictly (Proposition 4.1); (ii) given a pair (h, α), a point $\mathbf{x}_* \in \text{Int}(\mathcal{S})$ is an equilibrium of the unfiltered system if and only if it is an equilibrium of the filtered system (Proposition 4.2); (iii) if a point $\mathbf{x}_* \in \partial\mathcal{S}$ is an equilibrium of the unfiltered system, it is also an equilibrium of the filtered system (Proposition 4.3); and (iv) given two pairs (h_1, α_1) and (h_2, α_2), one has that $v_{(h_1,\alpha_1)}(\mathbf{x}_*) = v_{(h_2,\alpha_2)}(\mathbf{x}_*)$ for $\mathbf{x}_* \in \partial\mathcal{S}$ (Proposition 4.4). These results showcase to what extent the properties of equilibria of (10) are independent of the choice of the pair (h, α).

We note that the results presented in this section do not require $v(\mathbf{x})$ or $v_{(h,\alpha)}(\mathbf{x})$ to be locally Lipschitz, or the solutions of the filtered or unfiltered systems to be unique.

Proposition 4.1 (**Dynamics are Independent of CBF at Points where CBF Constraint is not Active**): *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a CBF of $\mathcal{S}$, and suppose that (h, α) is a pair satisfying Definition 2.1, and such that Assumptions 3.2 and 3.5 with (h, α). Let $\mathbf{x}_* \in \mathcal{S}$ and assume that one of the following holds:*

(i) $\nabla h(\mathbf{x}_*)^\top (f(\mathbf{x}_*) + g(\mathbf{x}_*)v_{(h,\alpha)}(\mathbf{x}_*)) + \alpha(h(\mathbf{x}_*)) > 0$;
(ii) $\nabla h(\mathbf{x}_*)^\top (f(\mathbf{x}_*) + g(\mathbf{x}_*)v(\mathbf{x}_*)) + \alpha(h(\mathbf{x}_*)) > 0$.

Then, there exists an open neighborhood $N_{\mathbf{x}_*}$ of $\mathbf{x}_*$ such that $v_{(h,\alpha)}(\mathbf{x}) = v(\mathbf{x})$ for all $\mathbf{x} \in N_{\mathbf{x}_*}$.

PROOF. We only prove the case when (i) holds and note that the argument for (ii) proceeds similarly. Since (h, α) satisfies Assumptions 3.2 and 3.5 [11, Theorem 5.3] ensures that $v_{(h,\alpha)}$ is continuous at $\mathbf{x}_*$. Moreover, since ∇h, f and g are continuous, there exists an open neighborhood $N_{\mathbf{x}_*}$ of $\mathbf{x}_*$ such that

$$\nabla h(\mathbf{x})^\top (f(\mathbf{x}) + g(\mathbf{x})v_{(h,\alpha)}(\mathbf{x})) + \alpha(h(\mathbf{x})) > 0 \tag{11}$$

for all $\mathbf{x} \in N_{\mathbf{x}_*}$. Now, since again (h, α) satisfies Assumptions 3.2 and 3.5, the optimizers of (7) satisfy the Karush–Kuhn–Tucker (KKT) conditions for all $\mathbf{x} \in N_{\mathbf{x}_*}$. Let $\lambda : N_{\mathbf{x}_*} \to \mathbb{R}^q$ be the Lagrange multiplier associated with the constraints $\ell_i(\mathbf{x}, \mathbf{u}, \boldsymbol{\delta}) \leq 0$, and let $\lambda_{q+1} : N_{\mathbf{x}_*} \to \mathbb{R}$ be the Lagrange multiplier associated with the constraint $F_{h,f}(\mathbf{x}) + \nabla h(\mathbf{x})^\top g(\mathbf{x})\mathbf{u} \geq 0$. Therefore, for all $\mathbf{x} \in N_{\mathbf{x}_*}$, the tuple $(v_{h,\alpha}(\mathbf{x}), \boldsymbol{\delta}_{(h,\alpha)}, \lambda(\mathbf{x}), \lambda_{q+1}(\mathbf{x}))$ is a KKT point of (7). Now, from (11), it follows that $\lambda_{q+1}(\mathbf{x}) = 0$ for all $\mathbf{x} \in N_{\mathbf{x}_*}$. This implies that the tuple $(v_{h,\alpha}(\mathbf{x}), \boldsymbol{\delta}_{(h,\alpha)}(\mathbf{x}), \lambda(\mathbf{x}))$ is a KKT point of problem (6) for all $\mathbf{x} \in N_{\mathbf{x}_*}$, and therefore $v(\mathbf{x}) = v_{(h,\alpha)}(\mathbf{x})$ for all $\mathbf{x} \in N_{\mathbf{x}_*}$. □

Proposition 4.1 shows that at points where the CBF constraint is not active, the filtered and unfiltered systems are equivalent. Next, we examine the effect of the CBF constraint on equilibria, starting with those in the interior on the safe set.

Proposition 4.2 (**Effect of CBF constraint on Interior Equilibria**): *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a CBF of $\mathcal{S}$, and suppose that (h, α) is a pair satisfying Definition 2.1, and Assumptions 3.2 and 3.5 hold with (h, α). Then, a point $\mathbf{x}_* \in Int(\mathcal{S})$ is an equilibrium of the unfiltered system* (5) *if and only if $\mathbf{x}_*$ is an equilibrium of the filtered system* (10)*, in which case there exists a neighborhood $N_{\mathbf{x}_*}$ of $\mathbf{x}_*$ such that $v_{(h,\alpha)}(\mathbf{x}) = v(\mathbf{x})$, for all $\mathbf{x} \in N_{\mathbf{x}_*}$.*

PROOF. If $\mathbf{x}_*$ is an equilibrium point of the filtered system (10), then $f(\mathbf{x}_*) + g(\mathbf{x}_*)v_{(h,\alpha)}(\mathbf{x}_*) = 0$. Therefore, from $\mathbf{x}_* \in \mathrm{Int}(\mathcal{S})$, it follows that $\nabla h(\mathbf{x}_*)^\top(f(\mathbf{x}_*) + g(\mathbf{x}_*)v_{(h,\alpha)}(\mathbf{x}_*)) + \alpha(h(\mathbf{x}_*)) > 0$. From Proposition 4.1, it follows that there exists a neighborhood $N_{\mathbf{x}_*}$ of $\mathbf{x}_*$ such that $v_{(h,\alpha)}(\mathbf{x}) = v(\mathbf{x})$ for all $\mathbf{x} \in N_{\mathbf{x}_*}$. This implies that $\mathbf{x}_*$ is an equilibrium of the unfiltered system. The reverse implication follows a similar argument. ☐

Proposition 4.2 shows that in the interior of the safe set, the equilibria of the filtered and unfiltered systems coincide, for any pair (h, α). In addition, the vector field in a neighborhood of these interior equilibria is the same. Consequently, undesirable equilibria can only appear on the boundary of the safe set, to which we turn our attention next.

Proposition 4.3 (**Effect of CBF Constraint on Boundary Equilibria**): *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a CBF of $\mathcal{S}$, and suppose that (h, α) is a pair satisfying Definition 2.1, and Assumptions 3.2 and 3.5 hold with (h, α). Then, if a point $\mathbf{x}_* \in \partial\mathcal{S}$ is an equilibrium of the unfiltered system* (5)*, it is also an equilibrium of the filtered system* (10)*.*

PROOF. Let $(v(\mathbf{x}_*), \delta(\mathbf{x}_*), \lambda(\mathbf{x}_*))$ be a KKT point of (6) at $\mathbf{x}_*$. Since $\mathbf{x}_*$ is an equilibrium point of (6), $f(\mathbf{x}_*) + g(\mathbf{x}_*)v(\mathbf{x}_*) = \mathbf{0}_n$. Let us now show that $(v(\mathbf{x}_*), \delta(\mathbf{x}_*), \lambda(\mathbf{x}_*), 0)$ is a KKT point of (7), from which it follows that $\mathbf{x}_*$ is also an equilibrium point of (7). The non-negativity of the Lagrange multipliers of $(v(\mathbf{x}_*), \delta(\mathbf{x}_*), \lambda(\mathbf{x}_*), 0)$ and the satisfaction of complementary slackness follows from the fact that $(v(\mathbf{x}_*), \delta(\mathbf{x}_*), \lambda(\mathbf{x}_*)$ is a KKT point of (6). Finally, the primal feasibility of $(v(\mathbf{x}_*), \delta(\mathbf{x}_*))$ follows from the fact that $(v(\mathbf{x}_*), \delta(\mathbf{x}_*))$ satisfies the constraints in (6) and since $\mathbf{x}_*$ is an equilibrium point, $\nabla h(\mathbf{x}_*)^\top(f(\mathbf{x}_*) + g(\mathbf{x}_*)v(\mathbf{x}_*)) + \alpha(h(\mathbf{x}_*)) = \alpha(h(\mathbf{x}_*)) = 0$. ☐

Proposition 4.3 shows that the addition of the CBF constraint in (7) does not affect the equilibria of the unfiltered system in the boundary of the safe set, again for any pair (h, α). However, the vector field in a neighborhood of these boundary equilibria might change due to the additional CBF constraint. Additionally, new equilibria might emerge in $\partial\mathcal{S}$, which will be undesirable by definition.

The following result establishes that, under some mild assumptions, the vector field in (10) evaluated at points in $\partial\mathcal{S}$ is independent of the choice of the pair (h, α).

Proposition 4.4 (**Dynamics in the Boundary are Independent of** (h, α) **Pair**): *Let $h_1 : \mathbb{R}^n \to \mathbb{R}$ and $h_2 : \mathbb{R}^n \to \mathbb{R}$ be CBFs of $\mathcal{S}$, and suppose that (h_1, α_1) and (h_2, α_2) are pairs satisfying Definition 2.1, and Assumptions 3.2 and 3.5 hold with (h_i, α_i), $\forall\ i = 1, 2$. Then, $v_{(h_1,\alpha_1)}(\mathbf{x}) = v_{(h_2,\alpha_2)}(\mathbf{x})$ for all $\mathbf{x} \in \partial\mathcal{S}$.*

PROOF. Since Assumption 3.2 holds and the pairs (h_1, α_1) and (h_2, α_2) satisfy Assumptions 3.2 and 3.5, $v_{(h_1,\alpha_1)}(\mathbf{x})$ and $v_{(h_2,\alpha_2)}(\mathbf{x})$ satisfy the KKT equations of (7) for (h_1, α_1) and (h_2, α_2), respectively.
Let $(v_{(h_1,\alpha_1)}(\mathbf{x}), \boldsymbol{\delta}_{(h_1,\alpha)}(\mathbf{x}), \lambda(\mathbf{x}), \lambda_{q+1}(\mathbf{x}))$ be a KKT point for (7) under (h_1, α_1). If $\mathbf{x} \in \partial\mathcal{S}$, we must have $h_1(\mathbf{x}) = h_2(\mathbf{x}) = 0$ and $\nabla h_1(\mathbf{x}) = \zeta_{(h_1,h_2)}(\mathbf{x})\nabla h_2(\mathbf{x})$, with $\zeta_{(h_1,h_2)}(\mathbf{x}) \in \mathbb{R}_{>0}$, from Lemma 2.2. This implies that

$$(v_{(h_1,\alpha_1)}(\mathbf{x}), \boldsymbol{\delta}_{(h_1,\alpha)}(\mathbf{x}), \lambda(\mathbf{x}), \lambda_{q+1}(\mathbf{x})/\zeta_{(h_1,h_2)}(\mathbf{x}))$$

is a KKT point of (7) under (h_2, α_2), and therefore $v_{(h_1,\alpha_1)}(\mathbf{x}) = v_{(h_2,\alpha_2)}(\mathbf{x})$. ☐

Since the vector field in (10) evaluated at points in $\partial\mathcal{S}$ is independent of the choice of pair (h, α), trajectories that stay in the boundary at all times are not affected by the choice of (h, α). Formally, we have the following result.

Corollary 4.5 (**Trajectories along the Boundary are Independent of CBF**): *Let $h_1 : \mathbb{R}^n \to \mathbb{R}$ and $h_2 : \mathbb{R}^n \to \mathbb{R}$ be CBFs of $\mathcal{S}$, and suppose that (h_1, α_1) and (h_2, α_2) are pairs satisfying Definition 2.1, and Assumptions 3.2 and 3.5 hold with (h_i, α_i), $\forall\ i = 1, 2$. Let $\tau \in (0, +\infty]$. Then, $\mathbf{y} : [0, \tau) \to \partial\mathcal{S}$ is a solution of the filtered system under the pair (h_1, α_1) if and only if it is a solution of the filtered system under the pair (h_2, α_2). In particular, given $\mathbf{x}_* \in \partial\mathcal{S}$, $\mathbf{x}_*$ is an equilibrium point of the filtered system under the pair (h_1, α_1) if and only if it is an equilibrium of the filtered system under the pair (h_2, α_2).*

PROOF. Since $\mathbf{y}(t) \in \partial\mathcal{S}$ for all $t \in [0, \tau)$, by Proposition 4.4 it follows that $v_{(h_1,\alpha_1)}(\mathbf{y}(t)) = v_{(h_2,\alpha_2)}(\mathbf{y}(t))$ for all $t \in [0, \tau)$. This implies that for all $t \in [0, \tau)$,

$$(f + gv_{(h_1,\alpha_1)})(\mathbf{y}(t)) = (f + gv_{(h_2,\alpha_2)})(\mathbf{y}(t)),$$

and therefore $\mathbf{y}$ is a solution of the filtered system under the pair (h_1, α_1) if and only if it is a solution of the filtered system under the pair (h_2, α_2). Finally, the last claim follows from the fact that if $\mathbf{x}_* \in \partial\mathcal{S}$ is an equilibrium point for the filtered system under pair (h_1, α_1) (resp. (h_2, α_2)), then $\mathbf{y}(t) = \mathbf{x}_*$ for all $t \geq 0$ is a valid solution of the filtered system under the pair (h_1, α_1) (resp. (h_2, α_2)). ☐

Corollary 4.5 implies that the number and location of limit cycles and equilibria (and, in particular, undesirable equilibria) in $\partial\mathcal{S}$ are independent of the pair (h, α).

Remark 4.6 (**Extension to Multiple Obstacles**): The results in this section can be extended to the case of multiple disjoint obstacles. In particular, assume that the unsafe set can be represented as $\mathcal{S}^c = \bigcup_{i=1}^r \mathcal{S}_i^c$ and $\overline{\mathcal{S}_i^c} \cap \overline{\mathcal{S}_j^c} = \varnothing$ for all $i \neq j$. Let $(h^{(i)}, \alpha^{(i)})$ be a pair satisfying Definition 2.1 for the set $\mathcal{S}_i$. Then, a generalization of (7) for the case with multiple CBF constraints reads as

$$\begin{bmatrix} v_{[r]}(\mathbf{x}) \\ \delta_{[r]}(\mathbf{x}) \end{bmatrix} := \arg \min_{[\mathbf{u};\boldsymbol{\delta}] \in \mathbb{R}^{m+m_0}} \ell(\mathbf{x}, \mathbf{u}, \boldsymbol{\delta}) \tag{12}$$
$$\text{s.t. } \ell_i(\mathbf{x}, \mathbf{u}, \boldsymbol{\delta}) \leq 0,\ \forall\ i = 1, \dots q$$
$$F_{h^{(j)},\alpha^{(j)},f}(\mathbf{x}) + \nabla h^{(j)}(\mathbf{x})^\top g(\mathbf{x})\mathbf{u} \geq 0,$$

$\forall\, j \in [r]$.

where $F_{h^{(j)},\alpha^{(j)},f}(\mathbf{x}) = \nabla h^{(j)}(\mathbf{x})^\top f(\mathbf{x}) + \alpha^{(j)}(h^{(j)}(\mathbf{x}))$. We assume that Slater's condition holds for (12) for all $\mathbf{x} \in \mathcal{S}$ and that $\nabla h^{(j)}(\mathbf{x})$ and $\alpha^{(j)}(\cdot)$ are continuously differentiable for all $j \in [r]$.

For equilibria on $\mathrm{int}(\mathcal{S})$, the general strategy consists in viewing the first $r-1$ CBF constraints as general constraints ℓ_i in (7), applying Proposition 4.1, and repeating the argument $r-1$ times to get a result similar to Proposition 4.2. For equilibria on $\partial\mathcal{S}$, given any $\mathbf{x}_* \in \partial\mathcal{S}$, one must have that there exists a unique $j \in [r]$ such that $\mathbf{x}_* \in \partial\mathcal{S}_j$ (since we assume that the obstacles are disjoint). One can then view the i-th (for $i = j$) CBF constraint as the CBF constraint in (7) and view the other (for $i \neq j$) CBF constraints as general constraints ℓ_i. By applying Proposition 4.1 to the $(h^{(i)}, \alpha^{(i)})$ pairs for $i \neq j$, we get that in a neighborhood of $\mathbf{x}_*$ the dynamics are independent of the pairs $(h^{(i)}, \alpha^{(i)})$ for $i \neq j$. Moreover, by Corollary 4.5 the existence of $\mathbf{x}_*$ is independent of the pair $(h^{(j)}, \alpha^{(j)})$. We omit the details due to space limitations. □

5 Impact of CBF Selection in CBF-CLF Controllers

In this section, we focus on the particular case of (7) that emerges when using CBF constraints in conjunction with CLF constraints. This corresponds to the CLF-CBF QP controller introduced in Example 3.3.

Our goal is to study how the stability properties of the corresponding closed-loop system depend on the choice of the pair (h, α). To achieve this, we first characterize the set of equilibria of the CLF-CBF QP controller (Proposition 5.4); next, we characterize the set of undesirable equilibria (Lemma 5.6); and finally, we provide an expression for the Jacobian evaluated at the undesirable equilibria that shows how it depends on the pair (h, α) (Proposition 5.10).

Consider the filtered system

$$\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})\bar{v}_{(h,\alpha)}(\mathbf{x}), \tag{13}$$

where $\bar{v}_{(h,\alpha)}$ is defined in (8), as well as its unfiltered version

$$\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})\bar{v}(\mathbf{x}), \tag{14}$$

with

$$\begin{bmatrix} \bar{v}(\mathbf{x}) \\ \bar{\delta}(\mathbf{x}) \end{bmatrix} = \arg \min_{[\mathbf{u};\delta] \in \mathbb{R}^{m+1}} \frac{1}{2}\|\boldsymbol{u}\|^2_{G(\mathbf{x})} + \frac{1}{2}p\delta^2 \tag{15}$$
$$\text{s.t. } F_{V,f}(\mathbf{x}) + \nabla V(\mathbf{x})^\top g(\mathbf{x})\mathbf{u} \leq \delta.$$

Note that (8) is a special case of (7). The following set of conditions on (8) imply that Assumption 3.2 holds.

Assumption 5.1 (**Smoothness Assumption for CLF-CBF-QP**): *For* (8)*, assume that:*

- *G is continuously differentiable and $G(\mathbf{x})$ is positive definite for all $\mathbf{x} \in \mathbb{R}^n$;*
- *$V(\mathbf{x})$ is twice continuously differentiable and $\beta(\cdot)$ is continuously differentiable;*
- *$h(\mathbf{x})$ is twice continuously differentiable and $\alpha(\cdot)$ is continuously differentiable.* □

We also make the following feasibility assumption.

Assumption 5.2 (**Strict Feasibility for CLF-CBF QP**): *For any $\mathbf{x} \in \mathcal{S}$, these exists $\mathbf{u}$ such that $\nabla h(\mathbf{x})^\top (f(\mathbf{x}) + g(\mathbf{x})\mathbf{u}) + \alpha(h(\mathbf{x})) > 0$.* □

The following result shows that this assumption is equivalent to Assumption 3.5 for the CLF-CBF QP problem.

Proposition 5.3 (**Equivalence between Constraint Qualifications and Strict Feasibility**): *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a CBF of $\mathcal{S}$ and let (h, α) be a pair satisfying Definition 2.1. Then, the following statements are equivalent:*

(i) *Assumption 5.2 holds;*
(ii) *$\|g(\mathbf{x})^\top \nabla h(\mathbf{x})\| \neq 0$ for all $\mathbf{x} \in \{\mathbf{x} \in \mathcal{S} : \nabla h(\mathbf{x})^\top f(\mathbf{x}) + \alpha(h(\mathbf{x})) = 0\}$;*
(iii) *Slater's condition holds for all $\mathbf{x} \in \mathcal{S}$ for* (8)*;*
(iv) *LICQ holds for all $(\mathbf{x}, \mathbf{u}, \delta) \in \mathcal{S} \times \mathbb{R}^{m+1}$ for* (8).

PROOF. *(i)* $\Longrightarrow$ *(ii)*: By contradiction, assume there exists $\hat{\mathbf{x}} \in \mathcal{S}$ such that $\|g(\hat{\mathbf{x}})^\top \nabla h(\hat{\mathbf{x}})\| = 0$ and $\nabla h(\hat{\mathbf{x}})^\top f(\hat{\mathbf{x}}) + \alpha(h(\hat{\mathbf{x}})) = 0$. This means that for all $\mathbf{u} \in \mathbb{R}^m$, it holds that $\nabla h(\hat{\mathbf{x}})^\top (f(\hat{\mathbf{x}}) + g(\hat{\mathbf{x}})\mathbf{u}) + \alpha(h(\hat{\mathbf{x}})) = 0$, which contradicts *(i)*.

(ii) $\Longrightarrow$ *(iii)*: Suppose that Slater's condition does not hold at $\hat{\mathbf{x}} \in \mathcal{S}$. As we can always pick δ large enough to ensure that the CLF constraint holds strictly, then $\nabla h(\hat{\mathbf{x}})^\top (f(\hat{\mathbf{x}}) + g(\hat{\mathbf{x}})\mathbf{u}) + \alpha(h(\hat{\mathbf{x}})) = 0$ for all $\mathbf{u} \in \mathbb{R}^m$. It follows that $\nabla h(\hat{\mathbf{x}})^\top g(\hat{\mathbf{x}}) = 0$ and $\nabla h(\hat{\mathbf{x}})^\top f(\hat{\mathbf{x}}) + \alpha(h(\hat{\mathbf{x}})) = 0$, which contradicts *(ii)*.

(iii) $\Longrightarrow$ *(i)*: Follows from the definition of Slater's condition.

(ii) $\Longrightarrow$ *(iv)*: Let $(\mathbf{x}, \mathbf{u}, \delta) \in \mathcal{S} \times \mathbb{R}^{m+1}$. First suppose that $F_{V,f}(\mathbf{x}) + \nabla V(\mathbf{x})^\top g(\mathbf{x})\mathbf{u} = \delta$ and $F_{h,f}(\mathbf{x}) + \nabla h(\mathbf{x})^\top (\mathbf{x}) g(\mathbf{x})\mathbf{u} > 0$. Then, LICQ holds at $(\mathbf{x}, \mathbf{u}, \delta)$, because $\begin{bmatrix} g(\mathbf{x})^\top \nabla V(\mathbf{x}) \\ -1 \end{bmatrix} \neq \mathbf{0}_{m+1}$. Second, suppose that $F_{V,f}(\mathbf{x}) + \nabla V(\mathbf{x})^\top g(\mathbf{x})\mathbf{u} < \delta$ and $F_{h,f}(\mathbf{x}) + \nabla h(\mathbf{x})^\top (\mathbf{x}) g(\mathbf{x})\mathbf{u} = 0$. By *(ii)*, it follows that $g(\mathbf{x})^\top \nabla h(\mathbf{x}) \neq \mathbf{0}_m$. Then, LICQ holds at $(\mathbf{x}, \mathbf{u}, \delta)$ since $\begin{bmatrix} g(\mathbf{x})^\top \nabla h(\mathbf{x}) \\ 0 \end{bmatrix} \neq \mathbf{0}_{m+1}$. Third, suppose that $F_{V,f}(\mathbf{x}) + \nabla V(\mathbf{x})^\top g(\mathbf{x})\mathbf{u} = \delta$ and $F_{h,f}(\mathbf{x}) + \nabla h(\mathbf{x})^\top g(\mathbf{x})\mathbf{u} = 0$. Again, by *(ii)*, it follows that $g(\mathbf{x})^\top \nabla h(\mathbf{x}) \neq \mathbf{0}_m$. Then, LICQ holds at $(\mathbf{x}, \mathbf{u}, \delta)$ because $\begin{bmatrix} g(\mathbf{x})^\top \nabla V(\mathbf{x}) \\ -1 \end{bmatrix}$ and $\begin{bmatrix} g(\mathbf{x})^\top \nabla h(\mathbf{x}) \\ 0 \end{bmatrix}$ are linearly independent for all $\mathbf{x} \in \mathcal{S}$. Finally, if $F_{V,f}(\mathbf{x}) + \nabla V(\mathbf{x})^\top g(\mathbf{x})\mathbf{u} <$

δ and $F_{h,f}(\mathbf{x}) + \nabla h(\mathbf{x})^\top(\mathbf{x})g(\mathbf{x})\mathbf{u} > 0$, LICQ holds at $(\mathbf{x}, \mathbf{u}, \delta)$ trivially.

(iv) $\implies$ *(ii)*: Suppose that *(ii)* does not hold, then there exist $\hat{\mathbf{x}} \in \mathcal{S}$ such that $\|g(\hat{\mathbf{x}})^\top \nabla h(\hat{\mathbf{x}})\| = 0$ and $\nabla h(\hat{\mathbf{x}})^\top f(\hat{\mathbf{x}}) + \alpha(h(\hat{\mathbf{x}})) = 0$. It follows that $F_{h,f}(\mathbf{x}) + \nabla h(\mathbf{x})^\top(\mathbf{x})g(\mathbf{x})\mathbf{u} = 0$ for all $\mathbf{u} \in \mathbb{R}^m$. Then, for any $\mathbf{u} \in \mathbb{R}^m$ and $\delta \in \mathbb{R}$, the CBF constraint is active at $(\hat{\mathbf{x}}, \mathbf{u}, \delta)$. Now, since the gradient of such CBF constraint w.r.t $[\mathbf{u}; \delta]$ is $\begin{bmatrix} g(\hat{\mathbf{x}})^\top \nabla h(\hat{\mathbf{x}}) \\ 0 \end{bmatrix} = \mathbf{0}_{m+1}$, LICQ does not hold at $(\hat{\mathbf{x}}, \mathbf{u}, \delta)$, hence reaching a contradiction. $\square$

Under Assumptions 5.1 and 5.2, using Proposition 5.3 and [17, Theorem 4.1], we conclude that $\bar{v}_{(h,\alpha)}(\mathbf{x})$ is locally Lipschitz at all $\mathbf{x} \in \mathcal{S}$. In addition, we note that Assumption 5.2 is weaker than the condition $g(\mathbf{x})^\top \nabla h(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in \mathcal{S}$, which is the assumption used in [3, Theorem 3] to ensure local Lipschitzness of the controller.

5.1 Characterization of Undesirable Equilibria

In this section, we study the undesirable equilibria of the closed-loop system (13) under a CLF-CBF QP controller. We start by providing a description of the equilibria of the system. In the results that follow, the dependence on $\mathbf{x}$ is dropped when convenient in order to simplify the notation. The following result is adapted from [20, Theorem 1], where the matrix G is not present, and its proof is therefore omitted for space reasons.

Proposition 5.4 (**Equilibria under CLF-CBF QP**): *Let (h, α) and (V, β) be pairs satisfying Definition 2.1 and Definition 2.7, respectively. Define $D : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ by $D(\mathbf{x}) = g(\mathbf{x})G^{-1}(\mathbf{x})g^\top(\mathbf{x})$. Then, the set of equilibrium points of* (13) *is given by $\mathcal{E} = \mathcal{E}_{int} \cup \mathcal{E}_{\partial\mathcal{S}}$, where*

$$\mathcal{E}_{int} := \left\{ \mathbf{x} \in \Omega^{clf}_{cbf} \cap \mathrm{int}(\mathcal{S}) \middle| \; f = p\beta(V)D\nabla V \right\},$$
$$\mathcal{E}_{\partial\mathcal{S}} := \left\{ \mathbf{x} \in \Omega^{clf}_{cbf} \cap \partial\mathcal{S} \mid f = \lambda_1 D\nabla V - \lambda_2 D\nabla h \right\},$$

with λ_1, λ_2, $\Omega^{clf}_{\underline{cbf}}$, and Ω^{clf}_{cbf} defined as

$$\lambda_1 := |\Delta|^{-1} \left(F_{V,f} \|\nabla h\|^2_D - F_{h,f} \nabla V^\top D \nabla h \right), \tag{16a}$$
$$\lambda_2 := |\Delta|^{-1} \left(F_{V,f} \nabla V^\top D \nabla h - F_{h,f} \left(\|\nabla V\|^2_D + \frac{1}{p} \right) \right), \tag{16b}$$
$$\Omega^{clf}_{\underline{cbf}} := \left\{ \mathbf{x} \in \mathbb{R}^n : \frac{\nabla V^\top D \nabla h F_{V,f}}{p^{-1} + \|\nabla V\|^2_D} < F_{h,f}, F_{V,f} \geq 0 \right\},$$
$$\Omega^{clf}_{cbf} := \left\{ \boldsymbol{x} \in \mathbb{R}^n : g^\top \nabla h \neq \mathbf{0}_m, \; \lambda_1 \geq 0, \; \lambda_2 \geq 0 \right\},$$

and where

$$|\Delta| = \left(p^{-1} + \|\nabla V\|^2_D \right) \|\nabla h\|^2_D - \left(\nabla V^\top D \nabla h \right)^2,$$

is the determinant of the matrix

$$\Delta := \begin{bmatrix} \|\nabla V\|^2_D + 1/p & -\nabla V^\top D \nabla h \\ -\nabla h^\top D \nabla V & \|\nabla h\|^2_D \end{bmatrix}. \tag{17}$$

Using Lemma 2.2, we can observe that the values of λ_1 and $\lambda_2 \nabla h$ at points in $\partial\mathcal{S}$ as well as the sets $\Omega^{\mathrm{clf}}_{\mathrm{cbf}} \cap \partial\mathcal{S}$ and $\Omega^{\mathrm{clf}}_{\underline{\mathrm{cbf}}} \cap \partial\mathcal{S}$ in Proposition 5.4 are independent of the pair (h, α).

Remark 5.5 (**Comparison with [16, 20]**): Proposition 5.4 is adapted from [20, Theorem 1] and is similar to [16, Theorem 1], which assumes that $g(\mathbf{x})$ is full rank. Specifically, the set $\mathcal{E}_{\partial\mathcal{S}}$ in Proposition 5.4 corresponds to $\mathcal{E}_{\partial\mathcal{S},2}$ in [20]. Under Assumption 5.2, the set $\mathcal{E}_{\partial\mathcal{S},1}$ in [20] is empty. Therefore, [20, Theorem 2] and [16, Theorem 1] are equivalent under Assumption 5.2. $\square$

The following result characterizes the set of undesirable equilibria under the CLF-CBF QP controller.

Lemma 5.6 (**Characterization of Undesirable Equilibria under CLF-CBF QP**): *Let (h, α) be a pair satisfying Definition 2.1 and Assumption 5.2, and let (V, β) be a pair satisfying Definition 2.7. Define the set $\hat{\mathcal{E}} := \{\mathbf{x} \in \Omega^{clf}_{cbf} \cap \partial\mathcal{S} \,|\, \lambda_1 > 0, \lambda_2 > 0, f = \lambda_1 D\nabla V - \lambda_2 D\nabla h\}$. Then,*

$$\mathcal{E}_{\partial\mathcal{S}} \setminus \hat{\mathcal{E}} = \left\{ x \in \Omega^{clf}_{cbf} \cap \partial\mathcal{S} \mid f = p\beta(V) D\nabla V \right\}.$$

Moreover, $\mathbf{x}_* \in \partial\mathcal{S}$ is an equilibrium of the unfiltered system (14) *if and only if $\mathbf{x}_* \in \mathcal{E}_{\partial\mathcal{S}} \setminus \hat{\mathcal{E}}$. In other words, $\hat{\mathcal{E}}$ is the collection of all undesirable equilibria.*

PROOF. First, consider points in $\mathcal{E}_{\partial\mathcal{S}}$ and such that $\lambda_1 = 0$. This implies that

$$\begin{aligned} 0 =& \lambda_1 = F_{h,f} \nabla V^\top D \nabla h - F_{V,f} \|\nabla h\|^2_D \\ =& \nabla h^\top f \nabla V^\top D \nabla h - (\nabla V^\top f + \beta(V)) \|\nabla h\|^2_D \\ =& -\lambda_2 (\nabla h^\top D \nabla h) \nabla V^\top D \nabla h \\ & - (-\lambda_2 \nabla V^\top D \nabla h + \beta(V)) \|\nabla h\|^2_D \\ =& -\beta(V) \left\| g^\top \nabla h \right\|^2_{G^{-1}} \end{aligned}$$

which is impossible, since $g^\top \nabla h \neq \mathbf{0}_m$ and $\beta(V) > 0$ on $\Omega^{\mathrm{clf}}_{\mathrm{cbf}}$. Thus, for points in $\mathcal{E}_{\partial\mathcal{S}}$, λ_1 can never be 0. Now consider points in $\mathcal{E}_{\partial\mathcal{S}}$ and such that $\lambda_2 = 0$. We have $\nabla V^\top f = \lambda_1 \nabla V^\top D \nabla V = \lambda_1 \|g^\top \nabla V\|^2_{G^{-1}}$ and

$$\begin{aligned} 0 =& \lambda_2 = F_{h,f} \left(\|\nabla V\|^2_D + p^{-1} \right) - \left(F_{V,f} \nabla V^\top D \nabla h \right) \\ =& \left(\frac{\lambda_1}{p} - \beta(V) \right) \nabla V^\top D \nabla h. \end{aligned}$$

Note that if $\nabla V^\top D \nabla h = 0$, we must have $F_{h,f}(\mathbf{x}) = 0$ by (16b). Then, by (16a) it follows that

$$\lambda_1 \left(p^{-1} + \|\nabla V\|^2_D \right) = F_{V,f}$$

$$=\lambda_1 \|\nabla V\|_D^2 + \beta(V),$$

which implies that $\lambda_1 = p\beta(V)$ and $f = \lambda_1 D\nabla V$. Finally, we note that if $\mathbf{x}_* \in \mathcal{E}_{\partial\mathcal{S}} \setminus \hat{\mathcal{E}}$, then the KKT point of (8) must be $(\bar{v}_{(h,\alpha)}(\mathbf{x}_*), \bar{\delta}(\mathbf{x}_*), \lambda_1, 0)$, which implies that $(v_{(h,\alpha)}(\mathbf{x}_*), \delta(\mathbf{x}_*), \lambda_1)$ a KKT point of (15) at $\mathbf{x}_*$ and it is also an equilibrium of (14). On the other hand, if $\mathbf{x}_*$ is an equilibrium of (15) and $(v(\mathbf{x}_*), \delta(\mathbf{x}_*), \lambda(\mathbf{x}_*))$ is a KKT point of (15) at $\mathbf{x}_*$, then, $(v(\mathbf{x}_*), \delta(\mathbf{x}_*), \lambda(\mathbf{x}_*), 0)$ is a KKT point of (8). It follows that $\mathbf{x}_* \in \mathcal{E}_{\partial\mathcal{S}}$ but $\mathbf{x}_* \notin \hat{\mathcal{E}}$. □

Since the set $\Omega^{\mathrm{clf}}_{\mathrm{cbf}} \cap \partial\mathcal{S}$ is independent of the pair (h, α), Lemma 5.6 implies that the sets $\mathcal{E}_{\partial\mathcal{S}}$ and $\hat{\mathcal{E}}$ are independent of the pair (h, α).

5.2 Stability of Undesirable Equilibria

In this section, we provide an explicit expression for the Jacobian of the filtered system evaluated at undesirable equilibria. We also show that the characteristic polynomial of this Jacobian for pairs (h_i, α_i), $i = 1, 2$ with CBFs that are equivalent is the same except for a single factor, which allows us to conclude that the stability properties of undesirable equilibria do not depend on the selected representative of the equivalence class.

To begin, we introduce a final assumption in this section, which ensures that the unfiltered system has no equilibria on the boundary of the safe set.

Assumption 5.7 (Unfiltered System has no Equilibria on the Boundary): *The set* $\{\mathbf{x} \in \partial\mathcal{S} : f(\mathbf{x}) + g(\mathbf{x})\bar{v}(\mathbf{x}) = 0\}$ *is empty, where* $\bar{v}(\mathbf{x})$ *is defined in* (15). □

By Lemma 5.6, Assumption 5.7 implies that $\mathcal{E}_{\partial\mathcal{S}} = \hat{\mathcal{E}}$. In what follows, most of our results are derived for the set of equilibria in $\hat{\mathcal{E}}$. Under some fairly general conditions, Assumption 5.7 is satisfied. For instance, it is satisfied if $\nabla V(\mathbf{x})^\top f(\mathbf{x}) < 0$ for all $\mathbf{x} \in \partial\mathcal{S}$. If there exists a controller $k : \mathbb{R}^n \to \mathbb{R}^m$ that stabilizes a point $\mathbf{x}_*$ for system (3), satisfies the CLF condition (4), and has $\partial\mathcal{S}$ in the region of attraction of $\mathbf{x}_*$, this assumption is satisfied by taking $\hat{f} = f + gk$ instead of f. Indeed, since V is a CLF, $\nabla V(\mathbf{x})^\top(f(\mathbf{x}) + g(\mathbf{x})k(\mathbf{x})) < 0$ for all $\mathbf{x} \in \partial\mathcal{S}$. The following result provides an explicit expression for the Jacobian evaluated at the points in $\hat{\mathcal{E}}$.

Proposition 5.8 (Jacobian Expression for Equilibria at the Boundary): *Let* (h, α) *and* (V, β) *be pairs satisfying Definition 2.1 and Definition 2.7, respectively. Let* (h, α) *and* (V, β) *satisfy Assumptions 5.1 and 5.2. Further, assume that* $D(\mathbf{x}) = g(\mathbf{x})G^{-1}(\mathbf{x})g(\mathbf{x})^\top$ *is a constant matrix and that Assumption 5.7 holds. Then, the Jacobian of* $f(\mathbf{x}) + g(\mathbf{x})\bar{v}_{(h,\alpha)}(\mathbf{x})$ *evaluated at* $\mathbf{x}_* \in \mathcal{E}_{\partial\mathcal{S}}$ *is*

$$\begin{aligned} J_{h,\alpha}\mid_{\mathbf{x}_*} = J_f &- \frac{\square_2}{|\Delta|}(\nabla h^\top J_f + \alpha'(0)\nabla h^\top) \qquad (18)\\ &- (\boldsymbol{I}_n - \frac{\square_2}{|\Delta|}\nabla h^\top)\lambda_1 D H_V \\ &- \frac{\square_1}{|\Delta|}(\nabla V^\top J_f + \beta'(V(\mathbf{x}_*))\nabla V^\top - \lambda_1 V^\top H_V) \\ &+ \left(\boldsymbol{I}_n - \frac{\square_2}{|\Delta|}\nabla h^\top - \frac{\square_1}{|\Delta|}\nabla V^\top\right)\lambda_2 D H_h, \end{aligned}$$

where J_f *is the Jacobian matrix of* $f(\mathbf{x})$, H_V, H_h *are the Hessians of* $V(\mathbf{x})$ *and* $h(\mathbf{x})$ *respectively,* $\square_1 := (\Delta_{22}D\nabla V + \Delta_{21}D\nabla h)$ *and* $\square_2 := (\Delta_{12}D\nabla V + \Delta_{11}D\nabla h)$, λ_1, λ_2 *are defined as in* (16a) *and* (16b), *and* Δ *is defined as* (17).

PROOF. To compute the Jacobian of $g(\mathbf{x})\bar{v}_{(h,\alpha)}(\mathbf{x})$ at $\mathbf{x}_* \in \mathcal{E}_{\partial\mathcal{S}}$, we first consider the KKT equations associated with $\bar{v}_{(h,\alpha)}(\mathbf{x})$, and then derive a system of equality equations for $g(\mathbf{x})\bar{v}_{(h,\alpha)}(\mathbf{x})$. The application of the Implicit Function Theorem then gives us an expression for the Jacobian of $g(\mathbf{x})\bar{v}_{(h,\alpha)}(\mathbf{x})$ at $\mathbf{x}_* \in \mathcal{E}_{\partial\mathcal{S}}$. The KKT equations for (8) read as follows

$$\begin{aligned} &G(\mathbf{x})\mathbf{u} + \lambda_1 g(\mathbf{x})^\top \nabla V(\mathbf{x}) - \lambda_2 g(\mathbf{x})^\top \nabla h(\mathbf{x}) = 0, \\ &p\delta - \lambda_1 = 0, \\ &\lambda_1 \left(F_{V,f}(\mathbf{x}) + \nabla V(\mathbf{x})^\top g(\mathbf{x})\mathbf{u} - \delta\right) = 0, \\ &\lambda_2 \left(F_{h,f}(\mathbf{x}) + \nabla h(\mathbf{x})^\top g(\mathbf{x})\mathbf{u}\right) = 0, \\ &\lambda_1 \geq 0, \quad \lambda_2 \geq 0, \\ &F_{V,f}(\mathbf{x}) + \nabla V(\mathbf{x})^\top g(\mathbf{x})\mathbf{u} - \delta \leq 0, \\ &F_{h,f}(\mathbf{x}) + \nabla h(\mathbf{x})^\top g(\mathbf{x})\mathbf{u} \geq 0. \end{aligned}$$

By Assumptions 5.1 and 5.2, there exist functions $\lambda_1 : \mathcal{S} \to \mathbb{R}_{\geq 0}$, $\lambda_2 : \mathcal{S} \to \mathbb{R}_{\geq 0}$ such that the tuple $(\bar{v}_{(h,\alpha)}(\mathbf{x}), \bar{\delta}(\mathbf{x}), \lambda_1(\mathbf{x}), \lambda_2(\mathbf{x}))$ is the only solution to the above KKT equations.

Next, let $\hat{v} : \mathcal{S} \to \mathbb{R}^n$ be defined as $\hat{v}_{(h,\alpha)}(\mathbf{x}) = g(\mathbf{x})\bar{v}_{(h,\alpha)}(\mathbf{x})$. Then, $(\hat{v}_{(h,\alpha)}(\mathbf{x}), \bar{\delta}(\mathbf{x}), \lambda_1(\mathbf{x}), \lambda_2(\mathbf{x}), \mathbf{x})$ is a solution of $\tilde{F}(\hat{\mathbf{u}}, \delta, \lambda_1, \lambda_2, \mathbf{x}) = \mathbf{0}_{n+3}$ for all $\mathbf{x} \in \mathcal{S}$, where $\tilde{F} : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n+3}$ is defined as

$$\tilde{F}(\hat{\mathbf{u}}, \delta, \lambda_1, \lambda_2, \mathbf{x}) := \begin{bmatrix} \hat{\mathbf{u}} + \lambda_1 D(\mathbf{x})\nabla V - \lambda_2 D(\mathbf{x})\nabla h \\ p\delta - \lambda_1 \\ \lambda_1 \left(F_{V,f}(\mathbf{x}) + \nabla V(\mathbf{x})^\top \hat{\mathbf{u}} - \delta\right) \\ \lambda_2 \left(F_{h,f}(\mathbf{x}) + \nabla h(\mathbf{x})^\top \hat{\mathbf{u}}\right) \end{bmatrix}.$$

Further note that for points $\mathbf{x}_* \in \hat{\mathcal{E}}$, the tuple

$$\mathbf{t}_{\mathbf{x}_*} = (\hat{v}_{(h,\alpha)}(\mathbf{x}_*), \bar{\delta}(\mathbf{x}_*), \lambda_1(\mathbf{x}_*), \lambda_2(\mathbf{x}_*))$$

satisfies the strict complementary slackness condition (cf. [10]). Hence, by [10, Theorem 2.1], $\tilde{F}$ is continuously differentiable at $\mathbf{t}_{\mathbf{x}_*}$. Next, let us show that the matrix

$$M_1 := \frac{\partial \tilde{F}}{\partial(\mathbf{u}, \delta, \lambda_1, \lambda_2)} \Big|_{\mathbf{t}_{\mathbf{x}_*}} \qquad (19)$$

is invertible. Indeed, since the Lagrange multipliers are strictly positive at points in $\hat{\mathcal{E}}$ and the strict complementary slackness condition holds for $\mathbf{t}_{\mathbf{x}_*}$,

$$M_1 = \begin{bmatrix} \mathbf{I}_n & 0 & D\nabla V & -D\nabla h \\ 0 & p & -1 & 0 \\ \lambda_1 \nabla V^\top & -\lambda_1 & 0 & 0 \\ \lambda_2 \nabla h^\top & 0 & 0 & 0 \end{bmatrix},$$

where all entries of M_1 are evaluated at $\mathbf{x}_*$. Define $M_1^{11} := \begin{bmatrix} \mathbf{I}_n & 0 \\ 0 & p \end{bmatrix}$, $M_1^{12} := \begin{bmatrix} D\nabla V & -D\nabla h \\ -1 & 0 \end{bmatrix}$, $M_1^{21} := \begin{bmatrix} \lambda_1 \nabla V^\top & -\lambda_1 \\ \lambda_2 \nabla h^\top & 0 \end{bmatrix}$, $M_1^{22} := \mathbf{0}_{2\times 2}$, and note that

$$M_1 := \begin{bmatrix} M_1^{11} & M_1^{12} \\ M_1^{21} & M_1^{22} \end{bmatrix}.$$

Observe also that M_1^{11} is invertible and its Schur complement is

$$M_c = M_1^{22} - M_1^{21}(M_1^{11})^{-1} M_1^{12} = \begin{bmatrix} -\lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \Delta,$$

where Δ is defined in (17) and evaluated at $\mathbf{x}_*$. By the Cauchy-Schwartz inequality, $|\nabla h^\top D \nabla V| \leq \|\nabla h\|_D \|\nabla V\|_D$ and hence $|\Delta| > 0$. Therefore, the determinant of M_c is equal to $-\lambda_1(\mathbf{x}_*)\lambda_2(\mathbf{x}_*)|\Delta(\mathbf{x}_*)| < 0$. This implies that M_1 is invertible.

Now, by the Implicit Function Theorem [19, Theorem 2-12] applied to $\tilde{F}(\mathbf{u}, \delta, \lambda_1, \lambda_2, \mathbf{x}) = \mathbf{0}_{n+3}$ at the point $\mathbf{t}_{\mathbf{x}_*}$, it follows that there exists a neighborhood $U_{\mathbf{x}_*}$ of $\mathbf{x}_*$ and unique continuously differentiable functions $\theta : U_{\mathbf{x}_*} \to \mathbb{R}^n$, $\delta^* : U_{\mathbf{x}_*} \to \mathbb{R}$, $\lambda_1^* : U_{\mathbf{x}_*} \to \mathbb{R}_{\geq 0}$ and $\lambda_2^* : U_{\mathbf{x}_*} \to \mathbb{R}_{\geq 0}$ such that

$$\tilde{F}(\theta(\mathbf{x}), \delta^*(\mathbf{x}), \lambda_1^*(\mathbf{x}), \lambda_2^*(\mathbf{x}), \mathbf{x}) = \mathbf{0}_{n+3}$$

for all $\mathbf{x} \in U_{\mathbf{x}_*}$. Moreover, it holds that

$$\frac{\partial}{\partial \mathbf{x}} \begin{bmatrix} \theta(\mathbf{x}) & \delta^*(\mathbf{x}) & \lambda_1^*(\mathbf{x}) & \lambda_2^*(\mathbf{x}) \end{bmatrix}^\top \Big|_{\mathbf{x}=\mathbf{x}_*} = -M_1^{-1} M_2, \quad (20)$$

where

$$M_2 := \frac{\partial \tilde{F}}{\partial \mathbf{x}}((\hat{v}_{(h,\alpha)}(\mathbf{x}_*), \bar{\delta}(\mathbf{x}_*), \lambda_1(\mathbf{x}_*), \lambda_2(\mathbf{x}_*), \mathbf{x}_*)).$$

After some computations, one can obtain

$$M_2 := \begin{bmatrix} \lambda_1 D H_V - \lambda_2 D H_h \\ 0 \\ \lambda_1(\nabla V^\top J_f + \beta'(V(\mathbf{x}_*))\nabla V^\top) \\ \lambda_2(\nabla h^\top J_f + \alpha'(0)\nabla h^\top) \end{bmatrix},$$

where we have used the fact that $\mathbf{x}_* \in \mathcal{E}_{\partial \mathcal{S}}$. For convenience, let us define

$$M_3 := \frac{-1}{|\Delta|} \left(\Delta_{22} D\nabla V + \Delta_{21} D \nabla h \right) \nabla V^\top + \frac{-1}{|\Delta|} \left(\Delta_{12} D \nabla V + \Delta_{11} D \nabla h \right) \nabla h^\top.$$

Since the solution of $\tilde{F}(\hat{\mathbf{u}}, \delta, \lambda_1, \lambda_2, \mathbf{x}) = \mathbf{0}_{n+3}$ is unique for all $\mathbf{x} \in U_{\mathbf{x}_*}$, it follows that $\hat{v}_{(h,\alpha)}(\mathbf{x}) = \theta(\mathbf{x})$ for all $\mathbf{x} \in U_{\mathbf{x}_*}$. Therefore the Jacobian of $f(\mathbf{x}) + g(\mathbf{x})\bar{v}_{(h,\alpha)}(\mathbf{x})$ evaluated at $\mathbf{x}_*$ is equal to $J_f(\mathbf{x}_*) + \frac{\partial \hat{v}_{(h,\alpha)}}{\partial \mathbf{x}}(\mathbf{x}_*)$. Using (20) and, after some lengthy computations, we get

$$\begin{aligned}
&\frac{\partial \hat{v}_{(h,\alpha)}}{\partial \mathbf{x}}(\mathbf{x}_*) = \frac{\partial \theta}{\partial \mathbf{x}}(\mathbf{x}_*) \\
&= -(\mathbf{I}_n + M_3)(\lambda_1 D H_V - \lambda_2 D H_h) \\
&- \frac{1}{|\Delta|} \left(\Delta_{22} D \nabla V + \Delta_{21} D \nabla h \right) \left(\nabla V^\top J_f + \beta'(V) \nabla V^\top \right) \\
&- \frac{1}{|\Delta|} \left(\Delta_{12} D \nabla V + \Delta_{11} D \nabla h \right) \left(\nabla h^\top J_f + \alpha'(0) \nabla h^\top \right) \\
&= -(\mathbf{I}_n - \frac{\square_2}{|\Delta|} \nabla h^\top) \lambda_1 D H_V \\
&- \frac{\square_2}{|\Delta|} (\nabla h^\top J_f + \alpha'(0) \nabla h^\top) \\
&- \frac{\square_1}{|\Delta|} (\nabla V^\top J_f + \beta'(V) \nabla V^\top - \lambda_1 V^\top H_V) \\
&+ \left(\mathbf{I}_n - \frac{\square_2}{|\Delta|} \nabla h^\top - \frac{\square_1}{|\Delta|} \nabla V^\top \right) \lambda_2 D H_h
\end{aligned}$$

where $\square_1$ and $\square_2$ are given in the statement. □

Remark 5.9 (On the Assumption that D is Constant): The assumption that $D(\mathbf{x}) = g(\mathbf{x}) G(\mathbf{x})^{-1} g(\mathbf{x})^\top$ is constant is satisfied for several classes of systems, including mechanical systems like the ones considered in [9, Section III.B]. For those systems, g can be written as $g(\mathbf{x}) = E\bar{g}(\mathbf{x})$, with $E = [\mathbf{0}_{m\times(n-m)}, I_{m\times m}]^\top$ and $\bar{g}(\mathbf{x}) \in \mathbb{R}^{m\times m}$ invertible for all $\mathbf{x}$. In this case, one can take $G(\mathbf{x}) = g(\mathbf{x})^\top g(\mathbf{x})$ to achieve $D = E(E^\top E)^{-1} E^\top = E E^\top$. □

Leveraging Proposition 5.8, the following result shows how the spectrum of the Jacobian evaluated at points in $\hat{\mathcal{E}}$ depends on the pair (h, α).

Proposition 5.10 (Jacobian at Equilibria on the Boundary of CLF-CBF QP as a Function of (h, α)): *Let (h_1, α_1), (h_2, α_2) be the two pairs satisfying Definition 2.1, and (V, β) be any pair satisfying Definition 2.7. Let Assumption 5.1 and 5.2 be satisfied with (h_1, α_1), (h_2, α_2) and (V, β). Further assume that $D(\mathbf{x}) = g(\mathbf{x}) G^{-1}(\mathbf{x}) g(\mathbf{x})^\top$ is a constant matrix and Assumption 5.7 holds. Let $\square_{1,h_1}$ and $\square_{1,h_2}$ denote the expression $\square_1$ in Proposition 5.8 defined with h_1 and h_2, respectively. Define Δ_{h_1}, Δ_{h_2}, λ_{h_1}, and λ_{h_2} similarly. Let also J_{h_1,α_1} be the Jacobian of $f + g\bar{v}_{(h_1,\alpha_1)}$, and J_{h_2,α_2} be defined similarly. For $\mathbf{x}_* \in \mathcal{E}_{\partial \mathcal{S}}$, it holds that*

$$\nabla h_1^\top \frac{\square_{1,h_1}}{|\Delta_{h_1}|} |_{\mathbf{x}_*} = 0, \quad \nabla h_1^\top \frac{\square_{2,h_1}}{|\Delta_{h_1}|} |_{\mathbf{x}_*} = 1, \quad (21a)$$

$$\nabla h_1(\mathbf{x}_*)^\top J_{h_1,\alpha_1} |_{\mathbf{x}_*} = -\alpha_1'(0) \nabla h_1(\mathbf{x}_*)^\top, \quad (21b)$$

$$\det(sI - J_{h_1,\alpha_1} |_{\mathbf{x}_*}) = (s + \alpha_1'(0)) \det(sI - M), \quad (21c)$$

where $M \in \mathbb{R}^{(n-1)\times(n-1)}$ only depends on f, g, G, h_1 V, and β (note that the expressions $\frac{\square_1}{|\Delta|}$, $\frac{\square_2}{|\Delta|} \nabla h^\top$, λ_1, $\lambda_2 \nabla h$

are independent of the pair (h,α) and therefore in what follows these expressions are denoted without sub-indexes). Furthermore, if $h_1 \overset{H}{\sim} h_2$, then

$$
\begin{aligned}
&J_{h_2,\alpha_2}\mid_{\mathbf{x}_*} - J_{h_1,\alpha_1}\mid_{\mathbf{x}_*} \\
&= \Big[-(\alpha_2'(0)-\alpha_1'(0))\frac{\square_2}{|\Delta|}\nabla h^\top \\
&+ \left(\boldsymbol{I}_n - \frac{\square_2}{|\Delta|}\nabla h^\top - \frac{\square_1}{|\Delta|}\nabla V^\top\right)\lambda_{2,h_2} D\nabla h_1 \tilde{\zeta}^\top \\
&+ \left(\boldsymbol{I}_n - \frac{\square_2}{|\Delta|}\nabla h^\top - \frac{\square_1}{|\Delta|}\nabla V^\top\right)\lambda_{2,h_2} D\tilde{\zeta}\nabla h_1^\top \Big]\Big|_{\mathbf{x}_*}
\end{aligned} \tag{22a}
$$

and

$$
\frac{\det(sI - J_{h_2,\alpha_2}\mid_{\mathbf{x}_*})}{s+\alpha_2'(0)} = \frac{\det(sI - J_{h_1,\alpha_1}\mid_{\mathbf{x}_*})}{s+\alpha_1'(0)}. \tag{22b}
$$

PROOF. First, we note that (21a) follows from a direct computation. In turn, (21b) also follows by exploiting (21a). Next, let us show (21c). Since $\nabla h_1^\top \frac{\square_{2,h_1}}{|\Delta_{h_1}|}\mid_{\mathbf{x}_*\in\hat{\mathcal{E}}} = 1$, for $\mathbf{x}_* \in \hat{\mathcal{E}}$, then $\square_{2,h_1} \neq \mathbf{0}_n^\top$, and therefore there exist ξ_i, for $i=2,...,n$ such that $\left\{\frac{\square_{2,h_1}}{\|\square_{2,h_1}\|}\right\} \cup \{\xi_i\}_{i=2}^n$ is an orthonormal basis of $\mathbb{R}^n$ (since $\nabla h_1^\top \square_{2,h_1} \neq 0$, it also follows that $\{\nabla h_1(\mathbf{x}_*)\} \cup \{\xi_i\}_{i=2}^n$ is a basis of $\mathbb{R}^n$). Using the orthogonality of the basis, we have that for $i=2,...,n$,

$$
(J_{h_1,\alpha_1}\mid_{\mathbf{x}_*})^\top \xi_i = C_{h_1,\mathbf{x}_*}^\top \xi_i \tag{23}
$$

where $C_{h_1,\mathbf{x}_*} := J_f - \lambda_1 D H_V - \frac{\square_1}{|\Delta|}(\nabla V^\top J_f + \beta'(V)\nabla V^\top - \lambda_1 V^\top H_V) + \left(\mathbf{I}_n - \frac{\square_1}{|\Delta|}\nabla V^\top\right)\lambda_2 D H_{h_1}\mid_{\mathbf{x}_*}$. The right-hand side of (23) can be written as a linear combination of $\{\nabla h_1(\mathbf{x}_*)\} \cup \{\xi_i\}_{i=2}^n$ as

$$
C_{h_1,\mathbf{x}_*}^\top \xi_i = M_{1,i}\nabla h_1(\mathbf{x}_*) + \sum_{j=2}^n M_{j,i}\xi_i.
$$

Note that for all $i=2,\dots,n$, $j=1,\dots,n$, $M_{j,i}$ only depends on f, g, G, h_1, β and V, since $C_{h_1,\mathbf{x}_*}$ does not depend on α_1.

Next, define

$$
\begin{aligned}
T &:= \left[\nabla h_1(\mathbf{x}_*),\ \xi_2, \cdots, \xi_n\right] \in \mathbb{R}^{n\times n}, \\
\gamma_1 &:= \left[M_{1,2}\ M_{1,3}\ \cdots\ M_{1,n}\right] \in \mathbb{R}^{1\times(n-1)}, \\
M &:= \begin{bmatrix} M_{2,2} & M_{2,3} & \cdots & M_{2,n} \\ M_{3,2} & M_{3,3} & \cdots & M_{3,n} \\ \vdots & \vdots & \vdots & \vdots \\ M_{n,2} & M_{n,3} & \cdots & M_{n,n} \end{bmatrix} \in \mathbb{R}^{(n-1)\times(n-1)}.
\end{aligned}
$$

Using (21b), we can write

$$
T^{-1}(J_{h_1,\alpha_1}\mid_{\mathbf{x}_*})^\top T = \begin{bmatrix} -\alpha_1'(0) & \gamma_1 \\ \mathbb{0}_{n-1} & M \end{bmatrix}.
$$

This implies that $\det(sI - J_{h_1,\alpha_1}\mid_{\mathbf{x}_*\in\hat{\mathcal{E}}}) = (s+\alpha_1'(0))\det(sI-M)$, where $M \in \mathbb{R}^{(n-1)\times(n-1)}$ only depends on f, g, G, h_1, β and V, and therefore (21c) holds.

To show (22a), consider two equivalent CBFs and use the expression for the Jacobian in (18) to get for any $\mathbf{x}_* \in \hat{\mathcal{E}}$ that

$$
\begin{aligned}
&J_{h_2,\alpha_2}\mid_{\mathbf{x}_*} - J_{h_1,\alpha_1}\mid_{\mathbf{x}_*} \\
&= \Big[-(\alpha_2'(0)-\alpha_1'(0))\frac{\square_2}{|\Delta|}\nabla h^\top \\
&+ \left(\mathbf{I}_n - \frac{\square_2}{|\Delta|}\nabla h^\top - \frac{\square_1}{|\Delta|}\nabla V^\top\right)\lambda_{2,h_2} D H_{h_2} \\
&- \left(\mathbf{I}_n - \frac{\square_2}{|\Delta|}\nabla h^\top - \frac{\square_1}{|\Delta|}\nabla V^\top\right)\lambda_{2,h_1} D H_{h_1} \Big]\Big|_{\mathbf{x}_*}.
\end{aligned} \tag{24}
$$

Since $\nabla h_2(\mathbf{x}_*) = \zeta(\mathbf{x}_*)\nabla h_1(\mathbf{x}_*)$ and

$$
\begin{aligned}
H_{h_2}(\mathbf{x}_*) =& \nabla h_1(\mathbf{x}_*)\tilde{\zeta}(\mathbf{x}_*)^\top + \tilde{\zeta}(\mathbf{x}_*)\nabla h_1(\mathbf{x}_*)^\top \\
&+ \zeta(\mathbf{x}_*)H_{h_1}(\mathbf{x}_*),
\end{aligned}
$$

it follows that $\lambda_{2,h_2}(\mathbf{x}_*) = \lambda_{2,h_1}(\mathbf{x}_*)/\zeta(\mathbf{x}_*)$ and

$$
\begin{aligned}
&\left(\mathbf{I}_n - \frac{\square_2}{|\Delta|}\nabla h^\top - \frac{\square_1}{|\Delta|}\nabla V^\top\right)\lambda_{2,h_2} D H_{h_2} \\
&- \left(\mathbf{I}_n - \frac{\square_2}{|\Delta|}\nabla h^\top - \frac{\square_1}{|\Delta|}\nabla V^\top\right)\lambda_{2,h_1} D H_{h_1} \\
&= \left(\mathbf{I}_n - \frac{\square_2}{|\Delta|}\nabla h^\top - \frac{\square_1}{|\Delta|}\nabla V^\top\right)\lambda_{2,h_2} D\nabla h_1\tilde{\zeta}^\top \\
&\quad + \left(\mathbf{I}_n - \frac{\square_2}{|\Delta|}\nabla h^\top - \frac{\square_1}{|\Delta|}\nabla V^\top\right)\lambda_{2,h_2} D\tilde{\zeta}\nabla h_1^\top.
\end{aligned} \tag{25}
$$

The combination of (24) and (25) yields (22a).

Next, to show (22b), we compute $T^{-1}(J_{h_2,\alpha_2}\mid_{\mathbf{x}_*})^\top T$ using (22a). First, note that

$$
\begin{aligned}
\nabla h_1(\mathbf{x}_*)^\top J_{h_2,\alpha_2}\mid_{\mathbf{x}_*} &= \frac{1}{\zeta(\mathbf{x}_*)}\nabla h_2(\mathbf{x}_*)^\top J_{h_2,\alpha_2}\mid_{\mathbf{x}_*} \\
&= -\frac{1}{\zeta(\mathbf{x}_*)}\alpha_2'(0)\nabla h_2(\mathbf{x}_*)^\top = -\alpha_2'(0)\nabla h_1(\mathbf{x}_*)^\top.
\end{aligned}
$$

Next, we compute $\xi_i^\top J_{h_2,\alpha_2}\mid_{\mathbf{x}_*}$. Since we have already computed the value of $(J_{h_1,\alpha_1}\mid_{\mathbf{x}_*})^\top \xi_i$, cf. (23), it suffices to compute $(J_{h_2,\alpha_2}\mid_{\mathbf{x}_*} - J_{h_1,\alpha_1}\mid_{\mathbf{x}_*})^\top \xi_i$. Using (22a), this boils down to computing the multiplication between $\xi_i^\top$ and the three terms in (22a). Since $\xi_i^\top \square_{2,h_1} = 0$ by (21a), it follows that the multiplication between $\xi_i^\top$ and the first term is 0. The multiplication between $\xi_i^\top$ and the second term is

$$
\begin{aligned}
&\xi_i^\top \left(\mathbf{I}_n - \frac{\square_2}{|\Delta|}\nabla h^\top - \frac{\square_1}{|\Delta|}\nabla V^\top\right)\lambda_{2,h_2} D\nabla h_1\tilde{\zeta}^\top \\
&= \lambda_{2,h_2}\xi_i^\top D\nabla h_1\tilde{\zeta}^\top + \lambda_{2,h_2}\xi_i^\top \frac{\square_{1,h_1}}{|\Delta_{h_1}|}\Delta_{12,h_1}\tilde{\zeta}^\top \\
&= \lambda_{2,h_2}\xi_i^\top D\nabla h_1\tilde{\zeta}^\top
\end{aligned}
$$

$$+ \lambda_{2,h_2}\xi_i^\top \left(\frac{\square_{2,h_1}}{|\Delta_{h_1}|}\Delta_{22,h_1} - D\nabla h_1\right)\tilde{\zeta}^\top$$
$$=0,$$

where we have used the fact that

$$\begin{aligned}
&\frac{\square_{1,h_1}}{|\Delta_{h_1}|}\Delta_{12,h_1} \\
=&\frac{(\Delta_{22,h_1}D\nabla V + \Delta_{21,h_1}D\nabla h_1)}{\Delta_{11,h_1}\Delta_{22,h_1} - \Delta_{12,h_1}\Delta_{21,h_1}}\Delta_{12,h_1} \\
=&\frac{(\Delta_{12,h_1}\Delta_{22,h_1}D\nabla V + \Delta_{11,h_1}\Delta_{22,h_1}D\nabla h_1)}{\Delta_{11,h_1}\Delta_{22,h_1} - \Delta_{12,h_1}\Delta_{21,h_1}} - D\nabla h_1 \\
=&\Delta_{22,h_1}\frac{\square_{2,h_1}}{|\Delta_{h_1}|} - D\nabla h_1.
\end{aligned}$$

Therefore, for any $i = 2, ..., n$, we have

$$\begin{aligned}
&(J_{h_2,\alpha_2}\mid_{\mathbf{x}_*} - J_{h_1,\alpha_1}\mid_{\mathbf{x}_*})^\top \xi_i \\
&= \lambda_{2,h_2}\tilde{\zeta}^\top D\left(\mathbf{I}_n - \frac{\square_2}{|\Delta|}\nabla h^\top - \frac{\square_1}{|\Delta|}\nabla V^\top\right)\xi_i \nabla h_1(\mathbf{x}_*).
\end{aligned}$$

Hence

$$(J_{h_2,\alpha_2}\mid_{\mathbf{x}_*})^\top \xi_i = (M_{1,i} + \beta_{2,i})\nabla h_1 + \sum_{j=2}^{n} M_{j,i}\xi_i,$$

where $\beta_{2,i} := \lambda_{2,h_2}\tilde{\zeta}^\top D\left(\mathbf{I}_n - \frac{\square_2}{|\Delta|}\nabla h^\top - \frac{\square_1}{|\Delta|}\nabla V^\top\right)\xi_i$.

Now, define $\gamma_2 := \begin{bmatrix}\beta_{2,2} & \beta_{2,3} & \cdots & \beta_{2,n}\end{bmatrix} \in \mathbb{R}^{1\times(n-1)}$, and note that it holds that

$$T^{-1}(J_{h_2,\alpha_2}\mid_{\mathbf{x}_*})^\top T = \begin{bmatrix}-\alpha_2'(0) & \gamma_1 + \gamma_2 \\ \mathbb{0}_{n-1} & M\end{bmatrix}.$$

Therefore, $\det\left(sI - J_{h_2,\alpha_2}\mid_{\mathbf{x}_*\in\hat{\mathcal{E}}}\right) = (s + \alpha_2'(0))\det(sI - M) = \frac{s+\alpha_2'(0)}{s+\alpha_1'(0)}\det\left(sI - J_{h_1,\alpha_1}\mid_{\mathbf{x}_*\in\hat{\mathcal{E}}}\right)$, showing (22b). □

Proposition 5.10 shows that the stability properties of undesirable equilibria of the CLF-CBF QP are the same for equivalent CBFs. Recall that, given a CBF, Corollary 2.6 describes a way to compute a large set of CBFs that are equivalent to it.

Remark 5.11 (**Extension of CLF-CBF QP to Multiple Obstacles – Remark 4.6 cont'd**): We note that the version of the CLF-CBF QP with multiple CBF constraints is a special case of (12). Assume that the obstacles are disjoint as in Remark 4.6 and $\mathbf{x}_* \in \partial\mathcal{S}_j$ is any undesirable equilibrium of the resulting closed-loop system. By applying Proposition 4.1 to the $(h^{(i)}, \alpha^{(i)})$ pairs for $i \neq j$, we get that in a neighborhood of $\mathbf{x}_*$ the dynamics are independent of the pairs $(h^{(i)}, \alpha^{(i)})$ for $i \neq j$. Therefore, the controller obtained with this multi-CBF version of the CLF-CBF-QP is equal to $\bar{v}_{(h^{(j)},\alpha^{(j)})}(\mathbf{x})$ as defined in (8) in a neighborhood of $\mathbf{x}_*$, under Slater's condition and Assumption 5.1. Hence all the results for undesirable equilibria in this section apply to the multi-CBF version of CLF-CBF-QP in a neighborhood of $\mathbf{x}_*$. □

6 Impact of CBF Selection in Safety Filters

In this section, we study the particular case of (7) that emerges when filtering a nominal controller $k : \mathbb{R}^n \to \mathbb{R}^m$ with a CBF constraint. This corresponds to the safety filter introduced in Example 3.4. We are interested in studying how the stability properties of equilibrium points of (10), specifically the Jacobian matrix, depend on the choice of the pair (h, α).

Using the notation introduced in Example 3.4, $\breve{v}_{(h,\alpha)}$ can be written explicitly as, cf. [8, Theorem 1],

$$\breve{v}_{(h,\alpha)}(\mathbf{x}) = \begin{cases}\mathbf{0}_m, & \text{if } \eta(\mathbf{x}) \geq 0, \\ \bar{\mathbf{u}}(\mathbf{x}), & \text{if } \eta(\mathbf{x}) < 0,\end{cases} \tag{26}$$

where $\bar{\mathbf{u}}(\mathbf{x}) := -\frac{\eta(\mathbf{x})G(\mathbf{x})^{-1}g(\mathbf{x})^\top\nabla h(\mathbf{x})}{\|g(\mathbf{x})^\top\nabla h(\mathbf{x})\|^2_{G^{-1}(\mathbf{x})}}$ and $\eta(\mathbf{x}) := \nabla h(\mathbf{x})^\top(f(\mathbf{x}) + g(\mathbf{x})k(\mathbf{x})) + \alpha(h(\mathbf{x}))$. Under Assumption 5.2, which by Proposition 5.3 is equivalent to the fact that $\|g(\mathbf{x})^\top\nabla h(\mathbf{x})\| \neq 0$ for all $\mathbf{x}$ in $\{\mathbf{x} \in \mathcal{S} : \nabla h(\mathbf{x})^\top f(\mathbf{x}) + \alpha(h(\mathbf{x})) = 0\}$, it follows, using [2, Lemma III.2], that if f, g, k, G, ∇h and α are locally Lipschitz, then $\breve{v}_{(h,\alpha)}$ is also locally Lipschitz.

We note that [7, Lemma 1] provides a characterization of the equilibria set $\mathcal{E}_{\text{sf}}$ of the closed-loop system $\dot{\mathbf{x}} = \tilde{f}(\mathbf{x}) + g(\mathbf{x})\breve{v}_{(h,\alpha)}(\mathbf{x})$ (where $\tilde{f}(\mathbf{x}) := f(\mathbf{x}) + g(\mathbf{x})k(\mathbf{x})$) as $\mathcal{E}_{\text{sf}} = \hat{\mathcal{E}}_{\text{sf}} \cup \{\mathbf{x} : \tilde{f}(\mathbf{x}) = 0\}$,

$$\hat{\mathcal{E}}_{\text{sf}} := \{\mathbf{x} \in \partial\mathcal{S} : \exists\, \delta < 0 \text{ s.t. } \tilde{f}(\mathbf{x}) = \delta D(\mathbf{x})\nabla h(\mathbf{x})\},$$

and $D(\mathbf{x}) = g(\mathbf{x})G^{-1}(\mathbf{x})g(\mathbf{x})^\top$. Under the assumptions of Corollary 4.5, the set $\hat{\mathcal{E}}_{\text{sf}}$ is independent of the pair (h, α).

Let $\mathbf{x}^*$ be a desirable equilibrium of system $\dot{\mathbf{x}} = \tilde{f}(\mathbf{x})$ in Example 3.4. By Proposition 4.2, if $\mathbf{x}^* \in \{\mathbf{x} : \tilde{f}(\mathbf{x}) = 0\} \subset \text{Int}(\mathcal{S})$, the closed-loop system in a neighborhood of $\mathbf{x}^*$ is independent of the (h, α) pair. Hence, the stability properties of $\mathbf{x}^*$ are also independent of the (h, α) pair. Therefore, in what follows, we turn our attention to the equilibria on the boundary of the safe set.

We start by noting that, without the CLF constraint, the set $\hat{\mathcal{E}}$ defined in Lemma 5.6 reduces to $\hat{\mathcal{E}}_{\text{sf}}$. In such case, Assumption 5.7 reduces to the following one.

Assumption 6.1 (**Unfiltered System has no Equilibria on the Boundary**): *The set* $\{\mathbf{x} \in \partial\mathcal{S} : \tilde{f}(\mathbf{x}) = 0\}$ *is empty.*

Note that Assumption 6.1 trivially holds if k is a globally stabilizing controller. The following result shows how the spectrum of the Jacobian evaluated at points in $\hat{\mathcal{E}}_{\text{sf}}$ depends on the pair (h, α).

Proposition 6.2 (**Jacobian at Equilibria on the Boundary of Safety Filters as a Function of** (h, α)): *Let* (h_1, α_1), (h_2, α_2) *be two pairs satisfying Definition 2.1 and Assumption 5.2. Suppose that* $\tilde{f}$, α_1, α_2 *are differentiable, and* h_1, *and* h_2 *are twice continuously differentiable. Further assume that* $D := g(\mathbf{x})G^{-1}(\mathbf{x})g(\mathbf{x})^\top$ *is a constant matrix and Assumption 6.1 is satisfied. Let* $\mathbf{x}_* \in \hat{\mathcal{E}}_{sf}$. *Then, the Jacobian of* $\tilde{f}(\mathbf{x}) + g(\mathbf{x})\breve{v}_{(h_1,\alpha_1)}(\mathbf{x})$ *evaluated at* $\mathbf{x}_*$ *is*

$$J_{h_1,\alpha_1}|_{\mathbf{x}_*} = \Big[J_{\tilde{f}} - \frac{D\nabla h_1 \nabla h_1^\top}{\nabla h_1^\top D \nabla h_1}[J_{\tilde{f}} + \alpha_1'(0)\boldsymbol{I}_n] - \frac{D[\nabla h_1^\top \tilde{f} \boldsymbol{I}_n - \nabla h_1 \tilde{f}^\top]H_{h_1}}{\nabla h_1^\top D \nabla h_1} \Big]\Big|_{\mathbf{x}_*},$$

where $J_{\tilde{f}}(\mathbf{x}_*)$ *is the Jacobian matrix of* $\tilde{f}(\mathbf{x})$ *evaluated at* $\mathbf{x}_*$, *and* $H_{h_1}(\mathbf{x}_*)$ *is the Hessian of* $h_1(\mathbf{x})$ *evaluated at* $\mathbf{x}_*$. *As a consequence,*

$$\nabla h_1(\mathbf{x}_*)^\top J_{h_1,\alpha_1}|_{\mathbf{x}_*} = -\alpha_1'(0)\nabla h_1(\mathbf{x}_*)^\top$$
$$\det(sI - J_{h_1,\alpha_1}|_{\mathbf{x}_*}) = (s + \alpha_1'(0))\det(sI - M),$$

where $M \in \mathbb{R}^{(n-1)\times(n-1)}$ *only depends on* $\tilde{f}$, g, G *and* h_1. *Furthermore, if* $h_1 \overset{H}{\sim} h_2$, *then*

$$\frac{\det(sI - J_{h_2,\alpha_2}|_{\mathbf{x}_*})}{s + \alpha_2'(0)} = \frac{\det(sI - J_{h_1,\alpha_1}|_{\mathbf{x}_*})}{s + \alpha_1'(0)}.$$

The proof follows by adapting the argument in Proposition 5.10 and is ommitted for space reasons. Proposition 6.2 shows that the stability properties under the safety filter of the undesirable equilibria are the same for all equivalent CBFs. Finally, we note that this result can be extended to the case where safety filters include multiple CBF constraints, similarly to Remark 5.11.

7 Numerical Simulation

This section illustrates our results in a numerical example. We also study qualitatively the dependence of other dynamical properties, like transient behavior or the size of the region of attraction of the desired equilibria, on the CBF pair. Consider the integrator dynamics $\dot{\mathbf{x}} = \mathbf{u}$ with nominal controller

$$k(\mathbf{x}) = \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix} \mathbf{x},$$

which globally asymptotically stabilizes the origin. There is an obstacle in the environment in the form of a ball centered at $(2, 0)$ of radius one, so let $\mathcal{S} = \{\mathbf{x} : \|\mathbf{x} - (2,0)^\top\|^2 - 1 \geq 0\}$ be the safe set. We represent this with two CBF pairs, given by $h_1(\mathbf{x}) := \|\mathbf{x} - (2,0)^\top\|^2 - 1$, $h_2(\mathbf{x}) := (\|\mathbf{x} - (5,1)^\top\|^2 + 1)h_1(\mathbf{x})$, $\alpha_1(s) := s$ and $\alpha_2(s) = 10s$. Note that $h_1 \overset{\mathrm{H}}{\sim} h_2$. In Figure 3, we apply the safety filter with CBF pairs (h_i, α_j), $i, j \in \{1, 2\}$ (here, $G(\mathbf{x}) = g(\mathbf{x})^\top g(\mathbf{x}) = \mathbf{I}_2$) to the unfiltered system $\dot{\mathbf{x}} = k(\mathbf{x})$.

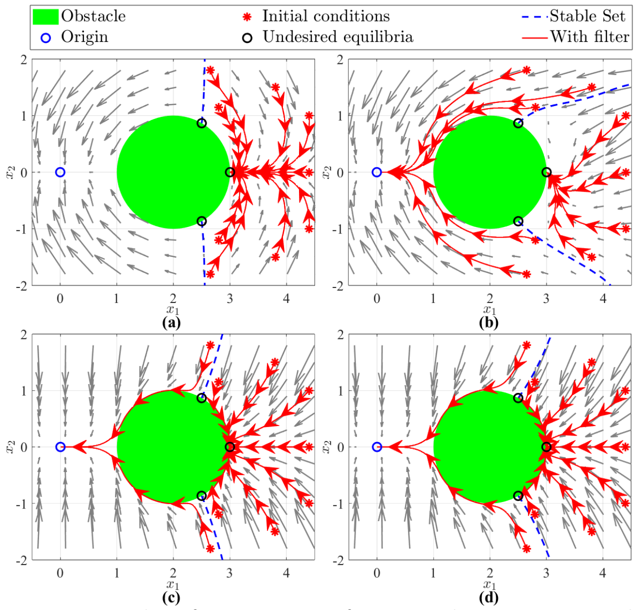

Fig. 3. Examples of trajectories of an LTI planar system with a safety filter for a circular obstacle; the figures show the vector fields, the undesired equilibria, the stable sets (stable manifolds) of the saddle equilibria, and the desired equilibrium (which is the origin). The plots are generated with different CBF pairs: (a): with CBF pair (h_1, α_1); (b): with CBF pair (h_2, α_1); (c): with CBF pair (h_1, α_2); (d) with CBF pair (h_2, α_2).

We compute the undesirable equilibria using the pair (h_1, α_1), as outlined in Section 6, and obtain three: $(\frac{5}{2}, \frac{\sqrt{3}}{2})^\top$, $(\frac{5}{2}, -\frac{\sqrt{3}}{2})^\top$ and $(3, 0)^\top$. By Corollary 4.5, these equilibria exist for any CBF pair. Furthermore, using the expression of the Jacobian in Proposition 6.2 with pair (h_1, α_1), we deduce that the equilibrium $(3, 0)^\top$ is asymptotically stable and the other two are saddle points. By Proposition 6.2, these stability properties hold for any pair for which the CBF is in the same equivalence class as h_1. This independence can be observed in Figure 3 (recall that $h_1 \overset{\mathrm{H}}{\sim} h_2$). In each of the plots, the trajectories converging to $(\frac{5}{2}, \frac{\sqrt{3}}{2})^\top$, $(\frac{5}{2}, -\frac{\sqrt{3}}{2})^\top$ and the points in the boundary of the obstacle with a value of x_1 smaller than or equal to $\frac{5}{2}$ constitute the boundary of the region of attraction of the origin.

Even though our results show that the number, location, and local stability properties of the equilibria remain unchanged under different CBF pairs, other dynamical properties of the filtered system may indeed change with the CBF pair. For example, Figure 3 shows that the size of the region of attraction of the origin is dependent on the CBF pair. We analyze the effect of the CBF and the extended class-$\mathcal{K}_\infty$ function on the region of attraction of the origin next. (i) If we fix $h = h_1$ (Figure 3(a),(c)), the region of attraction of the origin is larger for α_2 (which has a larger slope) than for α_1. However, if we fix $h = h_2$ (Figure 3(b),(d)), the region of attraction of the origin is larger for α_1 (which has a smaller slope) than for α_2. This suggests a complex dependence of the dynamical properties such as the region of attraction of the desired equilibrium on the CBF pair. (ii) If we fix $\alpha = \alpha_1$ (Figure 3(a),(b)), the choice of h significantly affects the region of attraction, whereas if we fix $\alpha = \alpha_2$ (Figure 3(c),(d)), the region of attraction remains almost the same for different h. We hypothesize that this is because the safety filter becomes inactive over a large region when the slope of α is large enough. Therefore, the filter can only be active at points close to $\partial\mathcal{S}$. However, by Proposition 6.2, the spectrum of the Jacobian evaluated at the undesirable equilibria remains unchanged with different h in the same equivalence class (like h_1 and h_2). All of this results in similar global dynamical behavior for different h when the slope of α is sufficiently large.

Finally, we describe the simulation setup considered in Figure 2. In Figure 2**(a)**, we use the same setting as [16, Fig 1]: $f(\mathbf{x}) = 0$, $g(\mathbf{x}) = \mathbf{I}_2$, nominal control $k(\mathbf{x}) = 0$, CLF $V(x) = \frac{1}{2}\lambda_1 x_1^2 + \frac{1}{2}\lambda_2 x_2^2, \lambda_1 = 6$ and $\lambda_2 = 1$ and CBF $h_1(x) = \frac{1}{2}\|x - x_c\|^2 - \frac{1}{2}1.5^2$, with $x_c = (0, 3)^\top$ and extended class-$\mathcal{K}_\infty$ function $\alpha_1(s) := s$. In Figure 2**(b)**, we change the CBF pair (h_1, α_1) to (h_2, α_2) and the rest parameters remain the same, where $h_2(\mathbf{x}) := (\|\mathbf{x} - (5, 1)^\top\|^2 + 1)h_1(\mathbf{x})$, and $\alpha_2(s) = 10s$. There are three undesirable equilibria: $(\sqrt{1.89}, 3.6)^\top$, $(-\sqrt{1.89}, 3.6)^\top$ and $(0, 4.5)^\top$, the last of which is asymptotically stable.

8 Conclusions

We have studied optimization-based control strategies for control-affine systems and investigated how the choice of the CBF impacts (desirable and undesirable) equilibria and the dynamical behavior of the resulting closed-loop system. We have shown that CBF-based constraints do not affect the number, location, and local stability properties of the equilibria in the interior of the safe set. We have also shown that undesirable equilibria only appear on the boundary of the safe set, and that their number and location do not depend of the choice of the CBF. For the specific case of safety filters and CLF-CBF QP controllers, we have shown that the stability properties of the closed-loop system are the same for all CBF pairs for which the corresponding CBFs are in the same equivalence class. In future work, we plan to extend the stability results to other CBF-based control designs and explore the interplay between the choice of CBF pair and the region of attraction of equilibria.

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Yiting Chen is a Ph.D. student in the Department of Electrical, Computer, and Energy Engineering at the University of Colorado Boulder. He obtained a Master's Degree in applied mathematics from The Chinese University of Hong Kong in 2022. His research interests span the areas of optimization, safe control, and learning.

Pol Mestres received the Bachelor's degree in mathematics and the Bachelor's degree in engineering physics from the Universitat Politècnica de Catalunya, Barcelona, Spain, in 2020, and the Master's degree in mechanical engineering in 2021 from the University of California, San Diego, La Jolla, CA, USA, where he is currently a Ph.D candidate. His research interests include safety-critical control, optimization-based controllers, distributed optimization and motion planning.

Jorge Cortés received the Licenciatura degree in mathematics from Universidad de Zaragoza, Spain, in 1997, and the Ph.D. degree in engineering mathematics from Universidad Carlos III de Madrid, Spain, in 2001. He held postdoctoral positions with the University of Twente, The Netherlands, and the University of Illinois at Urbana-Champaign, USA. He was an Assistant Professor with the Department of Applied Mathematics and Statistics, UC Santa Cruz, USA. He is a Professor in the Department of Mechanical and Aerospace Engineering, UC San Diego, USA. He is a Fellow of IEEE, SIAM, and IFAC. His research interests include distributed control and optimization, network science, nonsmooth analysis, reasoning and decision making under uncertainty, network neuroscience, and multi-agent coordination in robotic, power, and transportation networks.

Emiliano Dall'Anese is an Associate Professor in the Department of Electrical and Computer Engineering at Boston University, where he is also a faculty in the Division of Systems Engineering. He received the Ph.D. in Information Engineering from the Department of Information Engineering, University of Padova, Italy, in 2011. He was with the University of Minnesota as a postdoc (2011-2014), the National Renewable Energy Laboratory as a senior researcher (2014-2018), and with the Department of Electrical, Computer, and Energy Engineering at the University of Colorado Boulder as a faculty (2018-2024). His research interests span the areas of optimization, control, and learning; applications include power systems, transportation networks, and autonomous systems. He received the National Science Foundation CAREER Award in 2020, the IEEE PES Prize Paper Award in 2021, and the IEEE Transactions on Control of Network Systems Best Paper Award in 2023.