

Stability Analysis in Multi-Constraint Safety Filters for Linear Systems

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Abstract—Multi-constraint safety filters based on control barrier functions for linear systems with affine state constraints yield continuous piecewise-affine closed-loop dynamics and may introduce boundary equilibria and unstable active-set modes. Although they guarantee forward invariance, they can change nominal stability, and it remains unclear when unstable modes cause divergence versus bounded, convergent behavior. This paper develops a geometric framework to separate these cases: leveraging explicit active-set realizations, we show that equilibria associated with nonempty active sets lie on the corresponding constraint faces and that any unstable directions are tangent to those faces due to exponential enforcement of the active constraints. We characterize mode stability via a minimum-phase test, certify divergence under fixed active sets using recession cones, and derive tractable linear-matrix-inequality conditions for global exponential stability or boundedness using Lyapunov and LaSalle arguments.

I. INTRODUCTION

Safety-critical control requires feedback laws that achieve performance objectives while enforcing hard constraints, typically formalized as forward invariance of a safe set. Control barrier functions (CBFs) enable a practical “safety filter” architecture: a nominal stabilizing controller is minimally modified by solving a quadratic program (QP) that enforces CBF inequalities and guarantees invariance [1]–[3]. This separation of nominal performance design from safety enforcement is attractive in practice, but it also complicates analysis: the safety filter can alter stability and asymptotic behavior relative to the nominal closed loop, especially when multiple constraints are enforced simultaneously. This paper studies the following question for linear systems with affine state constraints: when can multi-constraint CBF-QP safety filters introduce boundary equilibria, unstable active-set modes, or unbounded trajectories, and when do trajectories remain bounded and converge to the desired equilibrium?

A growing literature has documented that CBF-QP safety filters may exhibit undesired dynamical behaviors beyond constraint satisfaction, including spurious equilibria, limit cycles, and divergence [4]–[6], but it offers limited structural explanation of when local instability leads to genuine divergence versus bounded long-term behavior. This gap is particularly evident even for linear systems with affine constraints. For the single-constraint case, a sharp characterization is available because the filtered closed loop reduces to a two-mode piecewise-affine system [7], [8], but these conclusions do not extend directly to multiple constraints, where many active sets can arise and trajectories may switch across

faces of the safe polyhedron. Related multi-constraint works have addressed feasibility and compatibility of simultaneous CBF inequalities [9], [10], but they do not address stability, equilibria, or divergence of the resulting closed-loop system.

In the multi-constraint regime, explicit and parametric QP results [11]–[13] imply a continuous piecewise-affine closed loop under standard regularity assumptions. In particular, for linear dynamics with affine constraints, the optimizer is piecewise affine and the state space admits a polyhedral active-set partition. General stability tools for piecewise-affine and switched systems provide useful sufficient conditions [14]–[16], but they do not provide a tailored dynamical understanding of multi-constraint CBF-QP safety filters, even though such filters are often used as modular wrappers and can reshape nominal stability and equilibria. This gap matters for reliable deployment because invariance alone does not preclude instability or undesirable asymptotic behavior.

This paper fills this gap via a geometric and invariant-set analysis that leverages explicit active-set realizations of multi-constraint CBF-QPs [17]. We first provide a spectral characterization of active-set modes and establish a minimum-phase interpretation of mode stability (Thm. 1, Cor. 1). We then characterize the equilibrium structure of the filtered closed loop, showing that equilibria associated with nonempty active sets lie on constraint boundaries (Prop. 1) and that instability under a fixed active set can only develop tangentially to the active constraints (Prop. 2, Cor. 3). Next, we provide a geometric certificate that distinguishes unstable modes that generate unbounded trajectories from those whose instability is suppressed by the polyhedral structure (Prop. 3). Finally, we derive tractable Lyapunov/LaSalle linear-matrix-inequality (LMI) conditions that certify global exponential stability or boundedness on the safe set (Thm. 2, Prop. 4).

II. PROBLEM SETUP AND CLOSED-LOOP STRUCTURE

Notation. The sets of real numbers and n -dimensional Euclidean space are denoted by $\mathbb{R}$ and $\mathbb{R}^n$. The interior and closure of a set $\mathcal{S} \subset \mathbb{R}^n$ are denoted by $\text{int}(\mathcal{S})$ and $\text{cl}(\mathcal{S})$. For vectors $x, y \in \mathbb{R}^n$, the inequalities $x \geq y$ and $x > y$ are understood componentwise. For symmetric matrices, positive definiteness and semidefiniteness are denoted by $P \succ 0$ and $P \succeq 0$. The identity matrix of dimension n is denoted by I_n , and by I when the dimension is clear from context. For a matrix $G \succ 0$, the weighted norm is defined as $\|z\|_G^2 := z^\top G z$. For scalars $a_1, \dots, a_k$, $\text{diag}(a_1, \dots, a_k)$ denotes the diagonal matrix with diagonal entries $a_1, \dots, a_k$.

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A. Linear System, Safety Constraints, and Safety Filter

We consider the linear control system

$$\dot{x} = Ax + Bu, \quad (1)$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the control input. A nominal state-feedback controller of the form

$$u = -Kx \quad (2)$$

is assumed to be given, with $A - BK$ Hurwitz.

Safety is specified by p affine state constraints

$$h_i(x) := c_i^\top x + d_i \geq 0, \quad i = 1, \dots, p, \quad (3)$$

where $c_i \in \mathbb{R}^n$ and $d_i > 0$. The corresponding safe set is

$$\mathcal{C} := \bigcap_{i=1}^p \{x \in \mathbb{R}^n : h_i(x) \geq 0\}. \quad (4)$$

The condition $d_i > 0$ for all $i \in [p]$ ensures that the origin lies in the interior of $\mathcal{C}$. Throughout the paper, we restrict attention to first-order (relative-degree-one) safety constraints, so that each constraint enters the dynamics affinely through the control input.

To enforce safety, we employ a CBF-based safety filter defined as the solution of the following QP:

$$\begin{aligned} u^*(x) = \arg \min_{u \in \mathbb{R}^m} & \frac{1}{2} \|u + Kx\|_G^2 \\ \text{s.t. } & c_i^\top (Ax + Bu) \geq -\alpha(c_i^\top x + d_i), \\ & i = 1, \dots, p, \end{aligned} \quad (5)$$

where $G \in \mathbb{R}^{m \times m}$ is symmetric positive definite and $\alpha > 0$ is a fixed scalar parameter.

Throughout the paper, we assume that the quadratic program (5) is feasible for all $x \in \mathcal{C}$ and that the resulting closed-loop system $\dot{x} = Ax + Bu^*(x)$ is well defined on $\mathcal{C}$. Under these assumptions, the safety filter renders $\mathcal{C}$ forward invariant and induces continuous piecewise-affine closed-loop dynamics.

B. Piecewise-Affine Closed-Loop Representation

We now describe the structure of the closed-loop dynamics induced by the safety filter (5). Although the control input is defined implicitly through a quadratic program, the resulting closed-loop system admits an explicit representation as a continuous piecewise-affine system on the safe set $\mathcal{C}$, which will be the basis for all subsequent analysis.

For any $x \in \mathcal{C}$, let $\mathcal{I}(x) \subseteq \{1, \dots, p\}$ denote the set of indices of constraints that are active at the optimizer of (5). For each $x \in \mathcal{C}$, (5) has a unique optimizer $u^*(x)$. Assume the active constraints have linearly independent u -gradients, i.e., $\text{rank}(B^\top C_{\mathcal{I}(x)}) = |\mathcal{I}(x)|$ for all $x \in \mathcal{C}$; hence $\mathcal{I}(x)$ is well defined and the active-set regions form a polyhedral partition of $\mathcal{C}$. This induces a partition of $\mathcal{C}$ into regions indexed by active sets,

$$\mathcal{R}_{\mathcal{I}} := \{x \in \mathcal{C} : \mathcal{I}(x) = \mathcal{I}\}, \quad (6)$$

where $\mathcal{I}$ ranges over all active sets that occur on $\mathcal{C}$. Under standard regularity conditions for strictly convex quadratic

programs with affine constraints, these regions are polyhedral and form a finite partition of $\mathcal{C}$. For each active set $\mathcal{I}$, define the closed cell $\bar{\mathcal{R}}_{\mathcal{I}} := \text{cl}(\mathcal{R}_{\mathcal{I}})$. In the affine-data setting, $\bar{\mathcal{R}}_{\mathcal{I}}$ is obtained by replacing strict inequalities in the KKT region characterization by non-strict ones.

On each region $\mathcal{R}_{\mathcal{I}}$, the optimizer $u^*(x)$ is affine in x , and the closed-loop dynamics take the affine form

$$\dot{x} = \tilde{A}_{\mathcal{I}}x + \tilde{b}_{\mathcal{I}}, \quad x \in \mathcal{R}_{\mathcal{I}}, \quad (7)$$

for matrices $\tilde{A}_{\mathcal{I}} \in \mathbb{R}^{n \times n}$ and vectors $\tilde{b}_{\mathcal{I}} \in \mathbb{R}^n$ that depend on the active set $\mathcal{I}$. When no constraint is active, $\mathcal{I} = \emptyset$, the safety filter coincides with the nominal controller, and

$$A_0 := \tilde{A}_{\emptyset} = A - BK, \quad \tilde{b}_{\emptyset} = 0. \quad (8)$$

We refer to the affine dynamics associated with any fixed active set $\mathcal{I}$ as a *mode* of the closed-loop system.

We next give explicit expressions for the matrices in (7). For each index set $\mathcal{I} \subseteq \{1, \dots, p\}$, let $C_{\mathcal{I}}$ denote the matrix whose columns are the vectors $\{c_i\}_{i \in \mathcal{I}}$, and let $d_{\mathcal{I}}$ denote the vector collecting the corresponding scalars $\{d_i\}_{i \in \mathcal{I}}$:

$$C_{\mathcal{I}} := [c_i]_{i \in \mathcal{I}} \in \mathbb{R}^{n \times |\mathcal{I}|}, \quad d_{\mathcal{I}} := [d_i]_{i \in \mathcal{I}} \in \mathbb{R}^{|\mathcal{I}|}. \quad (9)$$

Let

$$S_{\mathcal{I}} := C_{\mathcal{I}}^\top B G^{-1} B^\top C_{\mathcal{I}} \in \mathbb{R}^{|\mathcal{I}| \times |\mathcal{I}|}, \quad (10)$$

and assume $S_{\mathcal{I}} \succ 0$ (equivalently, $\text{rank}(B^\top C_{\mathcal{I}}) = |\mathcal{I}|$). Define

$$\Pi_{\mathcal{I}} := B G^{-1} B^\top C_{\mathcal{I}} S_{\mathcal{I}}^{-1} \in \mathbb{R}^{n \times |\mathcal{I}|}. \quad (11)$$

Then, on $\mathcal{R}_{\mathcal{I}}$, the closed-loop matrices are given by

$$\tilde{A}_{\mathcal{I}} = A_0 - \Pi_{\mathcal{I}} C_{\mathcal{I}}^\top (A_0 + \alpha I), \quad (12)$$

$$\tilde{b}_{\mathcal{I}} = -\alpha \Pi_{\mathcal{I}} d_{\mathcal{I}}. \quad (13)$$

Remark 1. Equations (12)–(13) follow from the KKT conditions of (5) by treating the constraints in $\mathcal{I}$ as equalities and solving for the corresponding Lagrange multipliers; see, e.g., [17].

C. Spectral Structure of Active-Set Modes

This subsection characterizes the spectrum of each affine mode matrix $\tilde{A}_{\mathcal{I}}$ in (12) in terms of the nominal closed-loop matrix $A_0 := A - BK$ and a square transfer matrix associated with the active constraints. This result will be used to interpret mode stability as a minimum-phase property. Define

$$F_{\mathcal{I}}(\lambda) := C_{\mathcal{I}}^\top (\lambda I - A_0)^{-1} B G^{-1} B^\top C_{\mathcal{I}}. \quad (14)$$

Theorem 1. Let $\mathcal{I} \neq \emptyset$ and suppose $S_{\mathcal{I}}$ in (10) is nonsingular. For any $\lambda \in \mathbb{C}$ such that $\lambda I - A_0$ is invertible,

$$\det(\lambda I - \tilde{A}_{\mathcal{I}}) = \det(\lambda I - A_0) \frac{(\lambda + \alpha)^{|\mathcal{I}|}}{\det(S_{\mathcal{I}})} \det(F_{\mathcal{I}}(\lambda)). \quad (15)$$

Proof. Write $\tilde{A}_{\mathcal{I}} = A_0 - U_{\mathcal{I}} V_{\mathcal{I}}^\top$ with $U_{\mathcal{I}} := \Pi_{\mathcal{I}}$ and $V_{\mathcal{I}}^\top := C_{\mathcal{I}}^\top (A_0 + \alpha I)$. By the matrix determinant lemma,

$$\det(\lambda I - \tilde{A}_{\mathcal{I}}) = \det(\lambda I - A_0) \det(I + V_{\mathcal{I}}^\top (\lambda I - A_0)^{-1} U_{\mathcal{I}}).$$

Using $(A_0 + \alpha I)(\lambda I - A_0)^{-1} = -I + (\lambda + \alpha)(\lambda I - A_0)^{-1}$ and $C_{\mathcal{I}}^\top \Pi_{\mathcal{I}} = I_{|\mathcal{I}|}$ yields

$$I + V_{\mathcal{I}}^\top (\lambda I - A_0)^{-1} U_{\mathcal{I}} = (\lambda + \alpha) C_{\mathcal{I}}^\top (\lambda I - A_0)^{-1} \Pi_{\mathcal{I}}.$$

Substituting $\Pi_{\mathcal{I}} = BG^{-1}B^\top C_{\mathcal{I}} S_{\mathcal{I}}^{-1}$ gives (15). $\square$

Corollary 1. Fix $\mathcal{I} \neq \emptyset$ and assume $S_{\mathcal{I}}$ in (10) is nonsingular. Define

$$M := BG^{-1}B^\top, \quad \mathcal{Z}_{\mathcal{I}}(\lambda) := \begin{bmatrix} \lambda I - A_0 & -MC_{\mathcal{I}} \\ C_{\mathcal{I}}^\top & 0 \end{bmatrix}.$$

Then, for all $\lambda \in \mathbb{C}$,

$$\det(\lambda I - \tilde{A}_{\mathcal{I}}) = \frac{(\lambda + \alpha)^{|\mathcal{I}|}}{\det(S_{\mathcal{I}})} \det(\mathcal{Z}_{\mathcal{I}}(\lambda)). \quad (16)$$

Consequently:

- 1) $\lambda = -\alpha$ is an eigenvalue of $\tilde{A}_{\mathcal{I}}$ with algebraic multiplicity at least $|\mathcal{I}|$;
- 2) for every $\lambda \neq -\alpha$ and λ different from any of the eigenvalues of A_0 , λ is an eigenvalue of $\tilde{A}_{\mathcal{I}}$ if and only if $\det(\mathcal{Z}_{\mathcal{I}}(\lambda)) = 0$.

Equivalently, every eigenvalue of $\tilde{A}_{\mathcal{I}}$ different from $-\alpha$ and the eigenvalues of A_0 , coincides with a finite invariant zero of the square system

$$\Sigma_{\mathcal{I}}: \quad \dot{x} = A_0 x + MC_{\mathcal{I}} v, \quad y = C_{\mathcal{I}}^\top x.$$

In particular, $\tilde{A}_{\mathcal{I}}$ is Hurwitz if and only if $\Sigma_{\mathcal{I}}$ is minimum phase, i.e., all finite invariant zeros of $\Sigma_{\mathcal{I}}$ lie in $\{\lambda \in \mathbb{C} : \Re(\lambda) < 0\}$.

Proof. By the Schur complement formula, $\det(\mathcal{Z}_{\mathcal{I}}(\lambda)) = \det(\lambda I - A_0) \det(F_{\mathcal{I}}(\lambda))$ for every λ such that $\lambda I - A_0$ is invertible. Combining this with (15) gives

$$\det(\lambda I - \tilde{A}_{\mathcal{I}}) = \frac{(\lambda + \alpha)^{|\mathcal{I}|}}{\det(S_{\mathcal{I}})} \det(\mathcal{Z}_{\mathcal{I}}(\lambda)).$$

Since both sides are polynomials in λ , this identity holds for all $\lambda \in \mathbb{C}$. The remaining claims follow immediately. $\square$

Remark 2. The factorization (15) is stated for $\lambda I - A_0$ invertible. The equivalent transmission-zero characterization in Cor. 1 avoids this restriction.

III. EQUILIBRIA AND ACTIVE-FACE GEOMETRY

In this section, we study the equilibrium structure of the piecewise-affine closed-loop dynamics (7). Since each active set defines a distinct affine mode, multiple constraints can generate several isolated equilibria or even continua on intersections of constraint boundaries.

A point $x^* \in \mathcal{C}$ is an equilibrium of the closed-loop dynamics (7) if and only if there exists an index set $\mathcal{I}$ such that

$$x^* \in \bar{\mathcal{R}}_{\mathcal{I}} \quad \text{and} \quad \tilde{A}_{\mathcal{I}} x^* + \tilde{b}_{\mathcal{I}} = 0; \quad (17)$$

see, e.g., [18], [19]. In particular, if $\tilde{A}_{\mathcal{I}}$ is nonsingular, then the corresponding candidate equilibrium is

$$x_{\mathcal{I}} := -\tilde{A}_{\mathcal{I}}^{-1} \tilde{b}_{\mathcal{I}},$$

which is an equilibrium of (7) only if $x_{\mathcal{I}} \in \mathcal{R}_{\mathcal{I}}$.

Throughout the paper, we take the desired equilibrium of the nominal closed-loop system as the reference point and, without loss of generality, place it at the origin by a translation of coordinates. All subsequent statements are made in these coordinates.

A. Active-Face Structure of Equilibria

We now establish a fundamental geometric property of equilibria of the closed-loop dynamics induced by multi-constraint safety filters. In contrast to unconstrained or single-constraint settings, any equilibrium associated with a nonempty active set lies on the boundary of the safe set, more precisely on the face determined by the active constraints. This property is specific to the piecewise-affine closed-loop structure induced by the safety filter and will play a central role in the analysis that follows. Recall that $\bar{\mathcal{R}}_{\mathcal{I}}$ denotes the closed cell associated with the active set $\mathcal{I}$.

Proposition 1. Let $\mathcal{I} \neq \emptyset$. If $x^* \in \bar{\mathcal{R}}_{\mathcal{I}}$ is an equilibrium of the closed-loop dynamics (7), then

$$C_{\mathcal{I}}^\top x^* + d_{\mathcal{I}} = 0, \quad (18)$$

where $C_{\mathcal{I}}$ and $d_{\mathcal{I}}$ are defined in (9).

Proof. Fix a nonempty active set $\mathcal{I}$ and let $x^* \in \bar{\mathcal{R}}_{\mathcal{I}}$ be a closed-loop equilibrium associated with this mode. Then $\tilde{A}_{\mathcal{I}} x^* + \tilde{b}_{\mathcal{I}} = 0$. Substituting (12)–(13) yields

$$\begin{aligned} 0 &= (A - BK)x^* - \Pi_{\mathcal{I}} C_{\mathcal{I}}^\top (A - BK + \alpha I)x^* - \alpha \Pi_{\mathcal{I}} d_{\mathcal{I}} \\ &= (A - BK)x^* - \Pi_{\mathcal{I}} \left(C_{\mathcal{I}}^\top (A - BK + \alpha I)x^* + \alpha d_{\mathcal{I}} \right). \end{aligned}$$

Premultiplying by $C_{\mathcal{I}}^\top$ gives

$$\begin{aligned} 0 &= C_{\mathcal{I}}^\top (A - BK)x^* \\ &\quad - C_{\mathcal{I}}^\top \Pi_{\mathcal{I}} \left(C_{\mathcal{I}}^\top (A - BK + \alpha I)x^* + \alpha d_{\mathcal{I}} \right). \end{aligned}$$

Using (11)–(10), we have

$$C_{\mathcal{I}}^\top \Pi_{\mathcal{I}} = C_{\mathcal{I}}^\top B G^{-1} B^\top C_{\mathcal{I}} S_{\mathcal{I}}^{-1} = I_{|\mathcal{I}|}.$$

Hence

$$\begin{aligned} 0 &= C_{\mathcal{I}}^\top (A - BK)x^* - \left(C_{\mathcal{I}}^\top (A - BK + \alpha I)x^* + \alpha d_{\mathcal{I}} \right) \\ &= -\alpha (C_{\mathcal{I}}^\top x^* + d_{\mathcal{I}}). \end{aligned}$$

Since $\alpha > 0$, it follows that $C_{\mathcal{I}}^\top x^* + d_{\mathcal{I}} = 0$. $\square$

Prop. 1 shows that any equilibrium associated with a nonempty active set $\mathcal{I}$ satisfies $C_{\mathcal{I}}^\top x^* + d_{\mathcal{I}} = 0$, and therefore lies on the constraint boundaries indexed by $\mathcal{I}$. This geometric fact will be used below to analyze motion along active faces and to explain why unstable modes need not produce unbounded trajectories.

Corollary 2. Let $\mathcal{I} \neq \emptyset$. If $\bar{\mathcal{R}}_{\mathcal{I}} \cap \partial \mathcal{C} = \emptyset$, then the closed-loop dynamics (7) have no equilibrium in $\bar{\mathcal{R}}_{\mathcal{I}}$. Consequently, every equilibrium of (7) belongs to $\{0\} \cup \partial \mathcal{C}$.

Proof. By Prop. 1, any equilibrium associated with a nonempty active set $\mathcal{I}$ must satisfy $C_{\mathcal{I}}^\top x + d_{\mathcal{I}} = 0$ and therefore lie on $\partial \mathcal{C}$. Hence, if $\bar{\mathcal{R}}_{\mathcal{I}}$ does not intersect $\partial \mathcal{C}$, it cannot contain a closed-loop equilibrium. $\square$

B. Tangency and Invariant Directions of Active Modes

In this subsection, we show that the affine dynamics associated with a fixed nonempty active set preserve the corresponding constraint equalities: once $C_{\mathcal{I}}^{\top}x + d_{\mathcal{I}} = 0$ holds, it continues to hold along trajectories of that mode. As a consequence, any instability of an active mode can only develop along directions tangent to these constraint boundaries. This property will be used in the sequel to distinguish between unstable yet bounded behavior and genuinely unbounded trajectories in the multi-constraint case.

Proposition 2. *Let $\mathcal{I} \neq \emptyset$, and let $x(\cdot)$ be a trajectory of the closed-loop dynamics (7) on an interval $[t_0, t_1]$ such that*

$$x(t) \in \mathcal{R}_{\mathcal{I}}, \quad \forall t \in [t_0, t_1].$$

Then:

$$\frac{d}{dt}(C_{\mathcal{I}}^{\top}x(t) + d_{\mathcal{I}}) = -\alpha(C_{\mathcal{I}}^{\top}x(t) + d_{\mathcal{I}}), \quad \forall t \in [t_0, t_1], \quad (19)$$

with $C_{\mathcal{I}}$ and $d_{\mathcal{I}}$ defined in (9). In particular, if $C_{\mathcal{I}}^{\top}x(t_0) + d_{\mathcal{I}} = 0$, then $C_{\mathcal{I}}^{\top}x(t) + d_{\mathcal{I}} = 0$, for all $t \in [t_0, t_1]$.

Proof. For $t \in [t_0, t_1]$, since $x(t) \in \mathcal{R}_{\mathcal{I}}$, the closed-loop dynamics (7) are given by $\dot{x}(t) = \tilde{A}_{\mathcal{I}}x(t) + \tilde{b}_{\mathcal{I}}$. Hence,

$$\frac{d}{dt}(C_{\mathcal{I}}^{\top}x + d_{\mathcal{I}}) = C_{\mathcal{I}}^{\top}\dot{x} = C_{\mathcal{I}}^{\top}(\tilde{A}_{\mathcal{I}}x + \tilde{b}_{\mathcal{I}}).$$

Using (12)–(13), we have $C_{\mathcal{I}}^{\top}(\tilde{A}_{\mathcal{I}}x + \tilde{b}_{\mathcal{I}}) = C_{\mathcal{I}}^{\top}(A - BK)x - C_{\mathcal{I}}^{\top}\Pi_{\mathcal{I}}(C_{\mathcal{I}}^{\top}(A - BK + \alpha I)x + \alpha d_{\mathcal{I}})$. From (11)–(10),

$$C_{\mathcal{I}}^{\top}\Pi_{\mathcal{I}} = C_{\mathcal{I}}^{\top}BG^{-1}B^{\top}C_{\mathcal{I}}S_{\mathcal{I}}^{-1} = I_{|\mathcal{I}|}.$$

Substituting gives

$$C_{\mathcal{I}}^{\top}(\tilde{A}_{\mathcal{I}}x + \tilde{b}_{\mathcal{I}}) = -\alpha(C_{\mathcal{I}}^{\top}x + d_{\mathcal{I}}),$$

which yields (19). The final claim follows by uniqueness of solutions of $\dot{z} = -\alpha z$ with $z(t_0) = 0$. $\square$

Prop. 2 shows that the intersection of the constraint boundaries indexed by $\mathcal{I}$ is forward invariant under the mode indexed by $\mathcal{I}$ (i.e., on any interval for which $x(t) \in \mathcal{R}_{\mathcal{I}}$). We next show that this invariance restricts the directions in which instability may arise.

Equation (19) admits the explicit solution

$$C_{\mathcal{I}}^{\top}x(t) + d_{\mathcal{I}} = e^{-\alpha(t-t_0)}(C_{\mathcal{I}}^{\top}x(t_0) + d_{\mathcal{I}}),$$

for any interval on which the active set $\mathcal{I}$ remains fixed. In particular, deviations from the constraint boundaries indexed by $\mathcal{I}$ decay exponentially at rate α . This shows that while instability may occur along directions tangent to the constraint boundaries, the component of the dynamics normal to those boundaries is uniformly contracting under a fixed active mode.

Corollary 3. *Let $\mathcal{I} \neq \emptyset$. If $v \in \mathbb{R}^n$ is an eigenvector of $\tilde{A}_{\mathcal{I}}$, defined in (12), associated with an eigenvalue $\lambda \neq -\alpha$, then*

$$C_{\mathcal{I}}^{\top}v = 0.$$

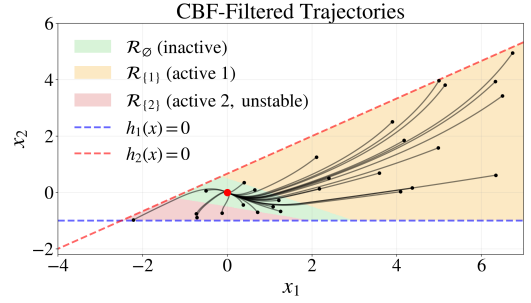


Fig. 1. Closed-loop trajectories for Example 1.

In particular, any eigenvector associated with an eigenvalue satisfying $\Re(\lambda) > 0$ is tangent to the constraint boundaries indexed by $\mathcal{I}$.

Proof. From (10), (11), and (12),

$$C_{\mathcal{I}}^{\top}\tilde{A}_{\mathcal{I}} = C_{\mathcal{I}}^{\top}A_0 - C_{\mathcal{I}}^{\top}\Pi_{\mathcal{I}}C_{\mathcal{I}}^{\top}(A_0 + \alpha I) = -\alpha C_{\mathcal{I}}^{\top}.$$

Let v satisfy $\tilde{A}_{\mathcal{I}}v = \lambda v$. Premultiplying by $C_{\mathcal{I}}^{\top}$ and using the identity above gives

$$\lambda C_{\mathcal{I}}^{\top}v = C_{\mathcal{I}}^{\top}\tilde{A}_{\mathcal{I}}v = -\alpha C_{\mathcal{I}}^{\top}v,$$

hence $(\lambda + \alpha)C_{\mathcal{I}}^{\top}v = 0$. If $\lambda \neq -\alpha$, then $C_{\mathcal{I}}^{\top}v = 0$. $\square$

Prop. 2, Cor. 3, and Prop. 1 show that, for any nonempty active set $\mathcal{I}$, the corresponding constraint boundaries are invariant under the mode (7) and any unstable eigendirection of $\tilde{A}_{\mathcal{I}}$ is tangent to them. Thus, in the multi-constraint case, mode instability alone does not imply unbounded trajectories.

The following example shows that, with multiple constraints, an active-set mode may be unstable while the closed-loop trajectories remain bounded.

Example 1. Consider (1) with $n = 2$, $m = 1$,

$$A = \text{diag}(-1, -2), \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and $u_{\text{nom}}(x) = -Kx$, where $K = \begin{bmatrix} 1 & 1 \end{bmatrix}$. Let $G = 1$ and $\alpha = 1$ in (5), and impose the constraints

$$h_1(x) = x_2 + 1 \geq 0, \quad h_2(x) = x_1 - \frac{3}{2}x_2 + 1 \geq 0.$$

For this system, the active-set mode matrix $\tilde{A}_{\{2\}}$ has a positive eigenvalue. Nevertheless, Fig. 1 shows that all trajectories starting in $\mathcal{C}$ remain bounded and converge to the origin. By Cor. 3, any unstable eigendirection of $\tilde{A}_{\{2\}}$ is tangent to the boundary $h_2(x) = 0$. However, trajectories leave $\mathcal{R}_{\{2\}}$ in finite time and switch modes, preventing unbounded growth under the fixed active set $\mathcal{I} = \{2\}$.

C. Divergence and Recession-Cone Characterization

Prop. 2 and Cor. 3 show that instability of an active mode $\tilde{A}_{\mathcal{I}}$ can only occur along directions tangent to the corresponding constraint boundaries. But tangential instability alone does not imply divergence: for divergence to occur while the active set remains fixed, the unstable direction must

stay inside both the active-set region and the safe set for arbitrarily large motion along that direction.

In the affine-data setting (linear dynamics, affine constraints, quadratic cost), each active-set region $\mathcal{R}_{\mathcal{I}}$ is a polyhedron. Because inactive constraints are typically defined by strict inequalities, we work with the closed cell $\bar{\mathcal{R}}_{\mathcal{I}}$. Since $\bar{\mathcal{R}}_{\mathcal{I}}$ is polyhedral, it admits a half-space representation

$$\bar{\mathcal{R}}_{\mathcal{I}} = \{x \in \mathbb{R}^n : H_{\mathcal{I}}x \leq h_{\mathcal{I}}\}. \quad (20)$$

Definition 1 (Recession Cone). For a closed convex set $\mathcal{P}$, its recession cone is

$$\text{rec}(\mathcal{P}) := \{v \in \mathbb{R}^n : x + \rho v \in \mathcal{P}, \forall x \in \mathcal{P}, \forall \rho \geq 0\}.$$

Geometrically, $\text{rec}(\mathcal{P})$ is the set of directions along which $\mathcal{P}$ extends to infinity.

The recession cone of a polyhedron admits a simple characterization. In particular, if $\mathcal{P} = \{x : Hx \leq h\}$, then (see, e.g., [20, Sec. 2.5])

$$\text{rec}(\mathcal{P}) = \{v \in \mathbb{R}^n : Hv \leq 0\}. \quad (21)$$

In particular, if $\bar{\mathcal{R}}_{\mathcal{I}}$ is bounded, then $\text{rec}(\bar{\mathcal{R}}_{\mathcal{I}}) = \{0\}$, so no nonzero direction can remain feasible for all scales under the fixed active-set inequalities.

We now give a divergence certificate for a fixed active set: if a mode has a real unstable eigenvector that is also a recession direction of the corresponding polyhedral region, then the closed-loop dynamics admit an unbounded trajectory that remains in that region.

Proposition 3. Let $\mathcal{I} \neq \emptyset$, and let $\bar{\mathcal{R}}_{\mathcal{I}}$ denote the closed cell associated with the active set $\mathcal{I}$. Assume that $\tilde{A}_{\mathcal{I}}$, given by (12), is invertible, and define $p_{\mathcal{I}} := -\tilde{A}_{\mathcal{I}}^{-1}\tilde{b}_{\mathcal{I}}$, where $\tilde{b}_{\mathcal{I}}$ is given by (13). Suppose that

- 1) $p_{\mathcal{I}} \in \bar{\mathcal{R}}_{\mathcal{I}}$,
- 2) $\tilde{A}_{\mathcal{I}}$ has a real eigenvalue $\lambda > 0$ with real eigenvector $v \neq 0$,
- 3) $v \in \text{rec}(\bar{\mathcal{R}}_{\mathcal{I}})$.

Then the closed-loop dynamics (7) admit an unbounded trajectory that remains in $\bar{\mathcal{R}}_{\mathcal{I}}$ for all $t \geq 0$.

Proof. Define $x(t) := p_{\mathcal{I}} + e^{\lambda t}v$ for $t \geq 0$. Since $\tilde{A}_{\mathcal{I}}p_{\mathcal{I}} + \tilde{b}_{\mathcal{I}} = 0$ and $\tilde{A}_{\mathcal{I}}v = \lambda v$, the trajectory $x(\cdot)$ satisfies $\dot{x} = \tilde{A}_{\mathcal{I}}x + \tilde{b}_{\mathcal{I}}$. Moreover, because $p_{\mathcal{I}} \in \bar{\mathcal{R}}_{\mathcal{I}}$ and $v \in \text{rec}(\bar{\mathcal{R}}_{\mathcal{I}})$, we have $p_{\mathcal{I}} + \rho v \in \bar{\mathcal{R}}_{\mathcal{I}}$ for all $\rho \geq 0$. Taking $\rho = e^{\lambda t}$ gives $x(t) \in \bar{\mathcal{R}}_{\mathcal{I}}$ for all $t \geq 0$. Since $\lambda > 0$ and $v \neq 0$, $\|x(t)\| \rightarrow \infty$ as $t \rightarrow \infty$. $\square$

Prop. 3 gives a sufficient certificate for divergence under a fixed active set: if $\tilde{A}_{\mathcal{I}}$ has a real unstable eigenvector in $\text{rec}(\bar{\mathcal{R}}_{\mathcal{I}})$, then the corresponding trajectory is unbounded and no mode switch occurs.

Finally, Prop. 3 gives a practical test for divergence under a fixed active set. In practice, one can: (i) compute the real unstable eigenpairs (λ, v) of $\tilde{A}_{\mathcal{I}}$ with $\lambda > 0$; (ii) check that the equilibrium satisfies $H_{\mathcal{I}}p_{\mathcal{I}} \leq h_{\mathcal{I}}$; and (iii) check the recession condition $H_{\mathcal{I}}v \leq 0$.

If the unstable subspace of $\tilde{A}_{\mathcal{I}}$ has dimension $r > 1$, a useful first step is to test whether it intersects the recession cone nontrivially. Let $U_{\mathcal{I}} \in \mathbb{R}^{n \times r}$ have columns spanning a real unstable invariant subspace of $\tilde{A}_{\mathcal{I}}$. Consider

$$\text{find } \eta \quad \text{s.t.} \quad H_{\mathcal{I}}U_{\mathcal{I}}\eta \leq 0, \quad \|\eta\|_1 = 1. \quad (22)$$

The normalization $\|\eta\|_1 = 1$ excludes the trivial solution and can be handled by the standard split $\eta = \eta^+ - \eta^-$ with $\eta^+, \eta^- \geq 0$. If (22) is feasible, then $v := U_{\mathcal{I}}\eta$ belongs to $\text{rec}(\bar{\mathcal{R}}_{\mathcal{I}})$ and gives a candidate divergence direction. Prop. 3 applies whenever this v is a real unstable eigenvector of $\tilde{A}_{\mathcal{I}}$, yielding the explicit trajectory $x(t) = p_{\mathcal{I}} + e^{\lambda t}v$.

These tests show why unstable active modes need not produce unbounded closed-loop behavior: if no unstable eigenvector lies in the recession cone of its region, then unbounded growth under a fixed active set is impossible. This does not by itself prove global boundedness, since divergence may still occur through switching, but it motivates the Lyapunov and invariance analysis in the next section.

Example 2. Consider (1) with

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad K = \begin{bmatrix} 4 & -1 \end{bmatrix},$$

with nominal controller $u_{\text{nom}}(x) = -Kx$, and let $\alpha = 1$ and $G = 1$ in (5). Impose the constraints

$$h_1(x) = x_2 + 1 \geq 0, \quad h_2(x) = -x_2 + 1 \geq 0.$$

For $\mathcal{I} = \{1\}$ and $\mathcal{I} = \{2\}$, the corresponding active modes are unstable, with unstable eigenspace $\text{span}\{e_1\}$ associated with the eigenvalue $\lambda = 1$. The corresponding mode equilibria are $p_{\{1\}} = [0 \ -1]^\top$, $p_{\{2\}} = [0 \ 1]^\top$, and satisfy $p_{\mathcal{I}} \in \bar{\mathcal{R}}_{\mathcal{I}}$ for $\mathcal{I} \in \{\{1\}, \{2\}\}$.

The test (22) selects the feasible recession direction $v = e_1$ for $\mathcal{I} = \{1\}$ and $v = -e_1$ for $\mathcal{I} = \{2\}$. Hence all assumptions of Prop. 3 hold. The resulting trajectories are

$$x_{\{1\}}(t) = p_{\{1\}} + e^t e_1 = \begin{bmatrix} e^t \\ -1 \end{bmatrix}, \quad x_{\{2\}}(t) = p_{\{2\}} - e^t e_1 = \begin{bmatrix} -e^t \\ 1 \end{bmatrix},$$

and remain in $\bar{\mathcal{R}}_{\mathcal{I}}$ for all $t \geq 0$. As shown in Fig. 2, trajectories in $\mathcal{R}_{\{1\}}$ diverge along $+e_1$ on the boundary $x_2 = -1$, while those in $\mathcal{R}_{\{2\}}$ diverge along $-e_1$ on the boundary $x_2 = 1$.

IV. BOUNDEDNESS AND INVARIANT-SET BEHAVIOR

The previous section showed that instability of an active mode need not imply divergence. Unbounded behavior requires an unstable direction that remains in the recession cone associated with $\bar{\mathcal{R}}_{\mathcal{I}}$. If this condition is not satisfied, and the trajectory associated with such unstable direction switches to other modes, boundedness might hold.

This section formalizes these observations using Lyapunov and invariance tools. We first give a strict common quadratic Lyapunov condition for global exponential stability, and then introduce less conservative region-wise Lyapunov conditions that exploit the polyhedral active-set partition. These results characterize the long-term behavior that can still be

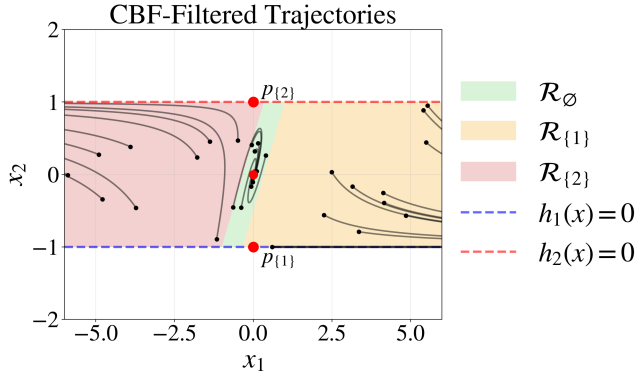


Fig. 2. Closed-loop trajectories for Example 2.

concluded when a strict common quadratic certificate is unavailable.

A. Global Exponential Stability via a CQLF

We now present a sufficient condition for global exponential stability (GES) of the origin based on the existence of a common quadratic Lyapunov function (CQLF) for all active-set modes of the closed-loop dynamics (7). Specifically, we seek a quadratic function $V(x) = x^\top Px$ that decreases strictly along every mode. Let

$$\mathcal{J} := \{\mathcal{I} \subseteq \{1, \dots, p\} : \mathcal{R}_{\mathcal{I}} \neq \emptyset\}$$

denote the set of active sets that occur on $\mathcal{C}$.

Theorem 2. Suppose there exist $P \succ 0$ and $\epsilon > 0$ such that

$$P\tilde{A}_{\mathcal{I}} + \tilde{A}_{\mathcal{I}}^\top P \preceq -\epsilon I, \quad \forall \mathcal{I} \in \mathcal{J}, \quad (23)$$

where $\tilde{A}_{\mathcal{I}}$ is given by (12). Then the origin is GES in $\mathcal{C}$ for the closed-loop dynamics (7).

Proof. Under the standing assumptions, the closed-loop dynamics (7) define a continuous piecewise-affine vector field on the forward-invariant set $\mathcal{C}$, with affine dynamics $f(x) = \tilde{A}_{\mathcal{I}}x + \tilde{b}_{\mathcal{I}}$ on each region $\mathcal{R}_{\mathcal{I}}$. Therefore, the hypotheses of [16, Thm. 1] apply on $\mathcal{C}$. Since (23) holds for all $\mathcal{I} \in \mathcal{J}$, [16, Thm. 1] yields exponential convergence of solutions on $\mathcal{C}$ in the quadratic metric induced by P . Since $u^*(0) = 0$, the origin is an equilibrium. By [16, Thm. 1], this means that the origin is GES in $\mathcal{C}$. $\square$

Corollary 4. Under the assumptions of Thm. 2, the origin is the unique equilibrium of the closed-loop dynamics (7) in $\mathcal{C}$.

Proof. Assume, for contradiction, that there exists an equilibrium $x^* \in \mathcal{C}$ of (7) with $x^* \neq 0$. Then the trajectory starting from x^* is constant, i.e., $x(t) \equiv x^*$ for all $t \geq 0$.

By Thm. 2, the origin is GES in $\mathcal{C}$. Hence there exist constants $c \geq 1$ and $\beta > 0$ such that every trajectory satisfies

$$\|x(t)\| \leq ce^{-\beta t} \|x(0)\|, \quad \forall t \geq 0.$$

Applying this bound to the trajectory with $x(0) = x^*$ gives

$$\|x^*\| = \|x(t)\| \leq ce^{-\beta t} \|x^*\|, \quad \forall t \geq 0,$$

which is impossible for $x^* \neq 0$ as $t \rightarrow \infty$. Therefore, no nonzero equilibrium of (7) exists in $\mathcal{C}$, and the origin is the unique equilibrium in $\mathcal{C}$. $\square$

Remark 3. The LMIs in (23) provide a tractable sufficient test for GES on $\mathcal{C}$, but the condition is conservative: even if every $\tilde{A}_{\mathcal{I}}$ is Hurwitz, a CQLF may not exist.

Remark 4. If the strict inequality in (23) is replaced by

$$P\tilde{A}_{\mathcal{I}} + \tilde{A}_{\mathcal{I}}^\top P \preceq 0, \quad \forall \mathcal{I} \in \mathcal{J},$$

then, under the standing assumptions that the closed-loop dynamics (7) are continuous piecewise affine on the convex forward-invariant set $\mathcal{C}$ and that the origin is an equilibrium, the line-segment argument of [21, Lemma 2] with $\alpha = 0$ implies $\dot{V}(x) \leq 0$ on $\mathcal{C}$ for $V(x) = x^\top Px$. Hence Thm. 3 yields boundedness and convergence to the largest invariant subset of $\{x \in \mathcal{C} : \dot{V}(x) = 0\}$.

B. Region-Wise Lyapunov Certificates

We now present a condition weaker than strict CQLF-based GES. Instead of requiring strict decay, we assume the existence of a common quadratic function that is nonincreasing along all modes.

Theorem 3. Let $V(x) := x^\top Px$ with $P \succ 0$. If

$$\nabla V(x)^\top (\tilde{A}_{\mathcal{I}}x + \tilde{b}_{\mathcal{I}}) \leq 0, \quad \forall x \in \overline{\mathcal{R}}_{\mathcal{I}}, \quad \forall \mathcal{I} \in \mathcal{J}, \quad (24)$$

where $\tilde{A}_{\mathcal{I}}$ and $\tilde{b}_{\mathcal{I}}$ are given by (12)–(13), then every trajectory of (7) starting in $\mathcal{C}$ is bounded and converges to the largest invariant subset of

$$\mathcal{E}_V := \{x \in \mathcal{C} : \dot{V}(x) = 0\}. \quad (25)$$

Proof. Since $\mathcal{C}$ is forward invariant and $V(x) = x^\top Px$ with $P \succ 0$, (24) implies that $V(x(t))$ is nonincreasing along trajectories of (7). Hence every trajectory starting in $\mathcal{C}$ remains in a bounded sublevel set of V , and is therefore bounded. Convergence to the largest invariant subset of $\mathcal{E}_V$ in (25) then follows from LaSalle's invariance principle for continuous piecewise-affine systems; see, e.g., [14], [19]. $\square$

Remark 5. Thm. 3 does not require each matrix $\tilde{A}_{\mathcal{I}}$ to be Hurwitz, and can therefore certify boundedness even when some active-set modes are unstable. Moreover, if the largest invariant subset of $\mathcal{E}_V$ in (25) is $\{0\}$, then LaSalle's invariance principle yields GES of the origin in $\mathcal{C}$, and hence uniqueness of the equilibrium in $\mathcal{C}$.

This condition should be contrasted with the strict CQLF requirement (23), which enforces strict decay of the linear part in every direction of $\mathbb{R}^n$, uniformly across modes. By contrast, (24) only requires Lyapunov decrease on each closed cell $\overline{\mathcal{R}}_{\mathcal{I}}$ and for the full affine vector field $\tilde{A}_{\mathcal{I}}x + \tilde{b}_{\mathcal{I}}$. These geometric restrictions can exclude the directions responsible for CQLF infeasibility, allowing a common quadratic Lyapunov function to certify stability of the piecewise-affine closed-loop dynamics defined by (7).

Under the hypotheses of Thm. 3, every trajectory starting in $\mathcal{C}$ is bounded. Moreover, if the active set is constant and

nonempty on some interval $[T, \infty)$, i.e., $x(t) \in \mathcal{R}_{\mathcal{I}}$ for all $t \geq T$ with $\mathcal{I} \neq \emptyset$, then Prop. 2 yields

$$C_{\mathcal{I}}^{\top} x(t) + d_{\mathcal{I}} = e^{-\alpha(t-T)} (C_{\mathcal{I}}^{\top} x(T) + d_{\mathcal{I}}) \rightarrow 0.$$

Thus, a persistent nonempty mode drives the trajectory toward the corresponding constraint face, while boundedness rules out divergence.

The next corollary gives a convenient, though conservative, matrix-only condition implying (24).

Corollary 5. *Let $V(x) := x^{\top} P x$ with $P \succ 0$. If*

$$P \tilde{A}_{\mathcal{I}} + \tilde{A}_{\mathcal{I}}^{\top} P \preceq 0, \quad \forall \mathcal{I} \in \mathcal{J},$$

then $\dot{V}(x) \leq 0$ for all $x \in \mathcal{C}$. Equivalently, (24) holds.

Proof. Because the closed-loop dynamics (7) define a continuous piecewise-affine vector field on $\mathcal{C}$, the line segment $[x_1, x_2] \subset \mathcal{C}$ intersects only finitely many switching hyperplanes. Repeating the line-segment argument of [21, Lemma 2] with $\alpha = 0$ yields

$$(x_1 - x_2)^{\top} P (\dot{x}_1 - \dot{x}_2) \leq 0, \quad \forall x_1, x_2 \in \mathcal{C},$$

where $\dot{x}_j$ denotes the value of the closed-loop vector field at x_j , $j = 1, 2$. Taking $x_2 = 0$ and using that the origin is an equilibrium gives $x^{\top} P \dot{x} \leq 0$ for all $x \in \mathcal{C}$. Hence

$$\dot{V}(x) = 2x^{\top} P \dot{x} \leq 0, \quad \forall x \in \mathcal{C},$$

which is (24). $\square$

The condition (24) can also be enforced directly on each region via a tractable LMI relaxation that exploits the polyhedral description of $\bar{\mathcal{R}}_{\mathcal{I}}$.

For $P = P^{\top} \in \mathbb{R}^{n \times n}$, define

$$Q_{\mathcal{I}}(P) := \tilde{A}_{\mathcal{I}}^{\top} P + P \tilde{A}_{\mathcal{I}}, \quad q_{\mathcal{I}}(P) := P \tilde{b}_{\mathcal{I}}, \quad (26)$$

where $\tilde{A}_{\mathcal{I}}$ and $\tilde{b}_{\mathcal{I}}$ are given by (12)–(13).

Proposition 4. *Let $V(x) := x^{\top} P x$ with $P = P^{\top}$, and suppose $\bar{\mathcal{R}}_{\mathcal{I}}$ is represented as in (20). Let $Q_{\mathcal{I}}(P)$ and $q_{\mathcal{I}}(P)$ be defined by (26). If there exist $P \succ 0$ and $\lambda_{\mathcal{I}} \geq 0$ such that*

$$\begin{bmatrix} Q_{\mathcal{I}}(P) & q_{\mathcal{I}}(P) - \frac{1}{2} H_{\mathcal{I}}^{\top} \lambda_{\mathcal{I}} \\ \star & \lambda_{\mathcal{I}}^{\top} h_{\mathcal{I}} \end{bmatrix} \preceq 0, \quad (27)$$

then $\dot{V}(x) \leq 0$ for all $x \in \bar{\mathcal{R}}_{\mathcal{I}}$. Consequently, if (27) holds for every $\mathcal{I} \in \mathcal{J}$, then (24) holds on $\mathcal{C}$.

Moreover, if there exist $P \succ 0$, $\varepsilon > 0$, and multipliers $\{\lambda_{\mathcal{I}} \geq 0\}_{\mathcal{I} \in \mathcal{J}}$ such that

$$\begin{bmatrix} Q_{\mathcal{I}}(P) + \varepsilon I & q_{\mathcal{I}}(P) - \frac{1}{2} H_{\mathcal{I}}^{\top} \lambda_{\mathcal{I}} \\ \star & \lambda_{\mathcal{I}}^{\top} h_{\mathcal{I}} \end{bmatrix} \preceq 0, \quad \forall \mathcal{I} \in \mathcal{J}, \quad (28)$$

then $\dot{V}(x) \leq -\varepsilon \|x\|^2$ on $\mathcal{C}$, and the origin is GES in $\mathcal{C}$ for the closed-loop dynamics (7).

Proof. Fix $\mathcal{I} \in \mathcal{J}$, and define

$$M_{\mathcal{I}} := \begin{bmatrix} Q_{\mathcal{I}}(P) & q_{\mathcal{I}}(P) - \frac{1}{2} H_{\mathcal{I}}^{\top} \lambda_{\mathcal{I}} \\ \star & \lambda_{\mathcal{I}}^{\top} h_{\mathcal{I}} \end{bmatrix}.$$

If $M_{\mathcal{I}} \preceq 0$, then for every $x \in \mathbb{R}^n$,

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^{\top} M_{\mathcal{I}} \begin{bmatrix} x \\ 1 \end{bmatrix} = x^{\top} Q_{\mathcal{I}}(P) x + 2q_{\mathcal{I}}(P)^{\top} x + \lambda_{\mathcal{I}}^{\top} (h_{\mathcal{I}} - H_{\mathcal{I}} x) \leq 0.$$

For $x \in \bar{\mathcal{R}}_{\mathcal{I}}$, represented as in (20), we have $h_{\mathcal{I}} - H_{\mathcal{I}} x \geq 0$. Since $\lambda_{\mathcal{I}} \geq 0$, the last term is nonnegative, and therefore

$$x^{\top} Q_{\mathcal{I}}(P) x + 2q_{\mathcal{I}}(P)^{\top} x \leq 0, \quad \forall x \in \bar{\mathcal{R}}_{\mathcal{I}}.$$

Using (26), this is exactly $\dot{V}(x) \leq 0$ on $\bar{\mathcal{R}}_{\mathcal{I}}$. If (27) holds for every $\mathcal{I} \in \mathcal{J}$, then $\dot{V}(x) \leq 0$ on $\mathcal{C}$, since

$$\mathcal{C} \subseteq \bigcup_{\mathcal{I} \in \mathcal{J}} \bar{\mathcal{R}}_{\mathcal{I}}.$$

For the exponential claim, replace $Q_{\mathcal{I}}(P)$ by $Q_{\mathcal{I}}(P) + \varepsilon I$ in $M_{\mathcal{I}}$. The same argument gives $\dot{V}(x) \leq -\varepsilon \|x\|^2$ on $\mathcal{C}$. Since

$$\lambda_{\min}(P) \|x\|^2 \leq V(x) \leq \lambda_{\max}(P) \|x\|^2,$$

it follows that

$$\dot{V}(x) \leq -(\varepsilon / \lambda_{\max}(P)) V(x),$$

hence V and $\|x\|$ decay exponentially. Therefore, the origin is GES in $\mathcal{C}$. $\square$

Condition (27) is affine in the decision variables $(P, \lambda_{\mathcal{I}})$ and yields a tractable LMI-based sufficient condition for (24). Although conservative relative to the exact region-wise inequality, it is typically less restrictive than strict CQLF conditions because it exploits both the polyhedral geometry of the active-set regions and the affine offsets $\tilde{b}_{\mathcal{I}}$.

V. ILLUSTRATIVE EXAMPLES AND DISCUSSION

We begin with an example in which the global CQLF test (23) succeeds, yielding GES via Thm. 2.

Example 3. Consider (1) with $K = 0$, $G = I_2$, and $\alpha = 1$, where

$$A = \begin{bmatrix} -1.5421 & 1.5125 \\ -0.2300 & -0.8362 \end{bmatrix}, \quad B = I_2.$$

Let the constraints be $h_1(x) = x_1 + 1 \geq 0$ and $h_2(x) = x_2 + 1 \geq 0$, so that

$$\mathcal{C} = \{x \in \mathbb{R}^2 : x_1 \geq -1, x_2 \geq -1\}$$

is unbounded. The resulting safety filter induces a continuous PWA closed-loop system with four active-set modes, whose linear parts are

$$\begin{aligned} \tilde{A}_{\emptyset} &= A, & \tilde{A}_{\{1\}} &= \begin{bmatrix} -1 & 0 \\ -0.2300 & -0.8362 \end{bmatrix}, \\ \tilde{A}_{\{1,2\}} &= -I_2, & \tilde{A}_{\{2\}} &= \begin{bmatrix} -1.5421 & 1.5125 \\ 0 & -1 \end{bmatrix}. \end{aligned} \quad (29)$$

Solving the strict CQLF LMIs in (23) yields a feasible matrix $P \succ 0$ and margin $\epsilon > 0$. It then follows from Thm. 2 that the origin is GES in $\mathcal{C}$, as illustrated in Fig. 3.

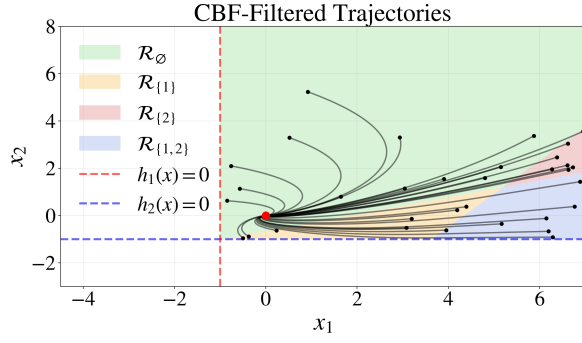


Fig. 3. Closed-loop trajectories for Example 3.

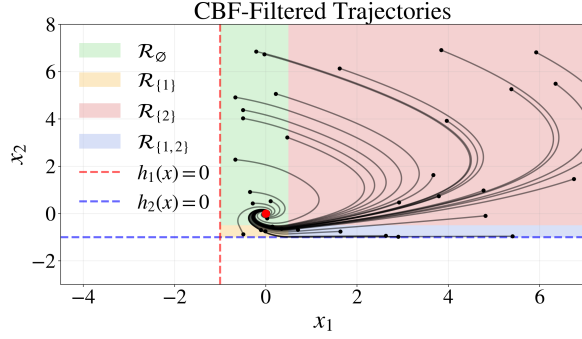


Fig. 4. Closed-loop trajectories for Example 4.

The next example shows that, even when (23) is infeasible, the region-wise LMI (27) in Prop. 4 can still certify global asymptotic stability of the origin.

Example 4. Consider (1) with $K = 0$, $G = I_2$, and $\alpha = 1$, where

$$A = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}, \quad B = I_2,$$

and constraints $h_1(x) = x_1 + 1 \geq 0$, $h_2(x) = x_2 + 1 \geq 0$. The safety filter induces a continuous piecewise-affine closed loop with four active-set modes.

The strict CQLF condition (23) in Thm. 2 is infeasible. Indeed, imposing it on the modes $\mathcal{I} = \{1\}$ and $\mathcal{I} = \{2\}$ yields the contradictory conditions $c > a$ and $a > c$ for

$$P = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \succ 0.$$

By contrast, the region-wise LMI (27) in Prop. 4 is feasible with the common choice $P = I_2$ and suitable multipliers $\lambda_{\mathcal{I}} \geq 0$, $\mathcal{I} \in \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. Hence (24) holds on $\mathcal{C}$ for $V(x) = x^\top x$. By Thm. 3, every trajectory starting in $\mathcal{C}$ is bounded and converges to the largest invariant subset of (25), as illustrated in Fig. 4. Moreover, a case-by-case analysis of the four active sets shows that $\dot{V}(x) < 0$, for all $x \in \mathcal{C} \setminus \{0\}$. Thus $\mathcal{E}_V = \{0\}$, and the origin is globally asymptotically stable in $\mathcal{C}$.

VI. CONCLUSIONS

This paper studied multi-constraint CBF-QP safety filters for linear systems with affine state constraints. We showed

that nonempty active sets can create boundary equilibria, that under a fixed active set instability can only develop tangentially to the active faces, and that divergence occurs only under an additional recession-cone condition. We also derived tractable minimum-phase, Lyapunov, and LMI-based tests for global exponential stability, boundedness, and asymptotic convergence. Future work will focus on less conservative certificates based on piecewise quadratic Lyapunov functions adapted to the polyhedral active-set structure.

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