Robinson's Counterexample and Regularity Properties of Optimization-Based Controllers

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Abstract

Robinson's counterexample shows that, even for relatively well-behaved parametric optimization problems, the corresponding optimizer might not be locally Lipschitz with respect to the parameter. In this brief note, we revisit this counterexample here motivated by the use of optimization-based controllers in closed-loop systems, where the parameter is the system state and the optimization variable is the input to the system. We show that controllers obtained from optimization problems whose objective and constraints have the same properties as those in Robinson's counterexample enjoy regularity properties that guarantee the existence (and in some cases, uniqueness) of solutions of the corresponding closed-loop system.

Keywords: Parametric optimization, optimization-based control, existence and uniqueness of solutions

1. Robinson's Counterexample

In [1], Robinson introduces the following parametric optimization problem: for $x = (x_1, x_2) \in \mathbb{R}^2$, consider

$$\min_{u \in \mathbb{R}^4} \frac{1}{2} u^T u \tag{1a}$$

$$\text{s.t. } A(x)u \geq b(x) \tag{1b}$$

where

$$A(x) = \begin{bmatrix} 0 & -1 & 1 & 0, \\ 0 & 1 & 1 & 0, \\ -1 & 0 & 1 & 0, \\ 1 & 0 & 1 & x_1 \end{bmatrix}, \quad b(x) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 + x_2 \end{bmatrix}.$$

Problem (1) is a quadratic program with strongly convex objective function, smooth objective function and constraints, and for which Slater's condition [2, Section 5.2.3] holds for every value of the parameter (this can be shown by noting that $\hat{u} = (0, 0, 2+|x_2|, 0)$ satisfies all constraints strictly). Despite these nice properties, the parametric solution of (1) is not locally Lipschitz at $(x_1, x_2) = (0, 0)$. Indeed, let $u^* : \mathbb{R}^2 \to \mathbb{R}^4$ be the parametric solution of (1) and $u_4^* : \mathbb{R}^2 \to \mathbb{R}$ denote its fourth component, which is given by

$$u_4^*(x_1, x_2) = \begin{cases} 0 & \text{if } x_2 \leq 0, \\ \frac{x_2}{x_1} & \text{if } x_2 \geq 0,\ x_1 \neq 0, \frac{x_1^2}{2} \geq x_2, \\ \frac{x_1(x_2+1)}{x_1^2+2} & \text{otherwise.} \end{cases}$$

The other components of u^* are continuously differentiable and therefore locally Lipschitz. However, if $p_{x_1} = (x_1, \frac{1}{2}x_1^2)$ and $q_{x_1} = (x_1, 0)$, we have

$$\frac{\| u_4^*(p_{x_1}) - u_4^*(q_{x_1}) \|}{\| p_{x_1} - q_{x_1} \|} = \frac{1}{x_1}.$$

Since x_1 can be taken to be arbitrarily small, this shows that u^* is not locally Lipschitz at the origin.

2. Parametric Optimization and Optimization-Based Controllers

The theory of parametric optimization [3, 4, 5] considers optimization problems that depend on a parameter and studies the regularity properties of the minimizers with respect to the parameter. Parametric optimization problems arise naturally in systems and control when designing

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optimization-based controllers, which are ubiquitous in numerous areas including safety-critical control [6], model predictive control [7, 8], and online feedback optimization [9, 10].

Given a system with state $x \in \mathbb{R}^n$, optimization-based controllers are feedback laws obtained by solving a problem of the form

$$\underset{u\in\mathbb{R}^m}{\operatorname{argmin}}\ f(x,u) \tag{2a}$$
$$\text{s.t. } g(x,u) \leq 0 \tag{2b}$$

with $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$. Note that the system state x acts as a parameter in (2). Assuming that the optimizer of (2) is unique for every $x \in \mathbb{R}^n$, this defines a function $u^* : \mathbb{R}^n \to \mathbb{R}^m$, mapping each state to the optimizer of (2). The flexibility of this approach allows to encode desirable goals for controller synthesis both in the cost function f (e.g., minimum control effort) and in the constraints g (e.g., prescribed decrease of a control Lyapunov function [11] or forward invariance of a set through a control barrier function [6]). Once synthesized, the controller u^* can be used to close the loop on the control system $\dot{x} = F(x,u)$ (here, $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz). Given the definition (2), the theory of parametric optimization can be brought to bear to characterize the regularity properties of u^*. These properties can then be used to certify existence and uniqueness of solutions of the closed-loop system

$$\dot{x} = F(x, u^*(x)). \tag{3}$$

For instance, if u^* is locally Lipschitz, then the right-hand side of (3) is locally Lipschitz too, and then the Picard-Lindelöf theorem [12, Theorem 2.2] guarantees existence and uniqueness of solutions. It is in this context that Robinson's counterexample is problematic, because it shows that, even for optimization problems defined by well-behaved data (including the widespread quadratic programs employed in the design of safe [6] and stabilizing [11] controllers), the resulting controller might not be locally Lipschitz. This has motivated the study [13, 14, 15] of additional conditions (which we make precise later) on the data of the optimization problem that guarantee local Lipschitzness and even stronger regularity properties of optimization-based controllers.

3. Paper Contributions

The note seeks to characterize the regularity properties enjoyed by the parametric optimizer of problems defined by objective and constraints with the same assumptions as in Robinson's counterexample. This is important as confusion may arise in the literature due to the loose use of terminology. Indeed, according to [4, Theorem 6.4], a parametric optimization problem whose data satisfies the properties of Robinson's counterexample has a locally Lipschitz minimizer! This apparent contradiction is rooted in different notions of Lipschitzness, which this note clarifies precisely. We show that, under the conditions of Robinson's counterexample, even though the parametric optimizer is not necessarily locally Lipschitz, it enjoys other desirable regularity properties. Moreover, we also show that under these regularity properties, the existence (and in some cases, uniqueness) of solutions of the closed-loop system obtained with the corresponding optimization-based controller are guaranteed. Finally, we conclude with an example that shows that, in general, these conditions are not enough to guarantee uniqueness of solutions of the closed-loop system, and stronger conditions are required.

4. Notions of Regularity of Functions

Throughout the note, we make use of the following notions of regularity of functions.

Definition 1. (Notions of Lipschitznes): *A function $f : \mathbb{R}^n \to \mathbb{R}^q$ is*

- point-Lipschitz at $x_0 \in \mathbb{R}^n$ *if there exists a neighborhood $\mathcal{U}$ of x_0 and a constant $L > 0$ such that*

$$\|f(x) - f(x_0)\| \leq L\|x - x_0\|, \quad \forall x \in \mathcal{U}. \tag{4}$$

- locally Lipschitz at $x_0 \in \mathbb{R}^n$ *if there exists a neighborhood $\tilde{\mathcal{U}}$ of x_0 and a constant $\tilde{L}$ such that*

$$\|f(x) - f(y)\| \leq \tilde{L}\|x - y\|, \quad \forall x, y \in \tilde{\mathcal{U}}. \tag{5}$$

The notion of point-Lipschitzness is used, for instance, in [4, Section 6.3] and called *Lipschitz stability*, without clearly acknowledging the difference with the notion of locally Lipschitzness. Studying point-Lipschitzness is natural in the context of parametric optimization, as one is normally interested in understanding the changes in the solution with respect to a *fixed* value of the parameter. Locally Lipschitz functions are point-Lipschitz, but the converse is not true. For instance, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x\sin(\frac{1}{x})$ is point-Lipschitz but not locally Lipschitz at the origin. Moreover, point-Lipschitz functions are continuous.

Definition 2. (Hölder property): *A function $f : \mathbb{R}^n \to \mathbb{R}^q$ has the* Hölder property at $x_0 \in \mathbb{R}^n$ *if there exists a neighborhood $\hat{\mathcal{U}}$ of x_0 and constants $C > 0$, $\alpha \in (0,1]$ such that*

$$\|f(x) - f(y)\| \leq C\|x - y\|^\alpha, \quad \forall x, y \in \hat{\mathcal{U}}. \tag{6}$$

Note that if f is locally Lipschitz at x_0 then it also has the Hölder property at x_0 but the converse is not true.

Definition 3. (Directionally differentiable function): *A function $f : \mathbb{R}^n \to \mathbb{R}$ is directionally differentiable if for any vector $v \in \mathbb{R}^n$, the limit*

$$\lim_{h\to 0} \frac{f(x + hv) - f(x)}{h}$$

exists. A vector-valued function is directionally differentiable if each of its components is directionally differentiable.

5. Regularity Properties of Parametric Optimizers under Assumptions of Robinson's Counterexample

We consider parametric optimization problems whose objective and constraints satisfy the same conditions as in Robinson's counterexample. The following result characterizes the regularity properties of the corresponding parametric optimizers.

Proposition 4. (Regularity Properties of Parametric Optimizer): *Suppose that f and g are twice continuously differentiable in $\mathbb{R}^n \times \mathbb{R}^m$. Further assume that given $x_0 \in \mathbb{R}^n$, $f(\cdot, x_0)$ is strongly convex, $g(\cdot, x_0)$ is convex and there exists $\hat{u} \in \mathbb{R}^m$ such that $g(\hat{u}, x_0) < 0$. Then,*

(i) *There exists a neighborhood $\mathcal{V}_{x_0}$ of x_0 such that u^* is point-Lipschitz at y for all $y \in \tilde{\mathcal{V}}_{x_0}$;*

(ii) *u^* has the Hölder property at x_0;*

(iii) *u^* is directionally differentiable at x_0.*

Proof. First we note that since $f(\cdot, x_0)$ is strongly convex and $g(\cdot, x_0)$ is convex for all x_0, $u^*(x_0)$ is unique and well-defined for all $x_0 \in \mathbb{R}^n$. To prove (i) we use [4, Theorem 6.4]. The fact that there exists $\hat{u} \in \mathbb{R}^m$ such that $g(\hat{u}, x_0) < 0$ implies that Slater's Condition (SC) holds. Hence, by [16, Prop. 5.39], since $g(\cdot, x_0)$ is convex, the Mangasarian-Fromovitz Constraint Qualification (MFCQ) holds at $(x_0, u^*(x_0))$. Furthermore, since $f(\cdot, x_0)$ is strongly convex and $g(\cdot, x_0)$ is convex, the second-order condition SOC2 [4, Definition 6.1] holds. All of this, together with the twice continuous differentiability of f and g imply, by [4, Theorem 6.4], that u^* is point-Lipschitz at x_0. Now, since g is continuous, there exists a neighborhood $\mathcal{V}_{x_0}$ of x_0 such that $g(\hat{u}, y) < 0$ for all $y \in \mathcal{V}_{x_0}$. By repeating the same argument, u^* is point-Lipschitz at y for all $y \in \mathcal{V}_{x_0}$. Now let us prove (ii). We use [17, Theorem 2.1], which gives a sufficient condition for the solution of a variational inequality to have the Hölder property. We first note that a constrained optimization problem of the form (2) can be posed as a variational inequality (cf. [18]). Since f is twice continuously differentiable and strongly convex, conditions (2.1) and (2.2) in [17, Theorem 2.1] hold. Moreover, since MFCQ holds at $(x_0, u^*(x_0))$ (because SC holds), by [19, Remark 3.6] the constraint set is pseudo-Lipschitzian [17, Definition 1.1]. All of this implies by [17, Theorem 2.1] that u^* has the Hölder property at x_0. Finally, (iii) follows from the fact that SC implies MFCQ and [20, Theorem 1]. □

In Proposition 4, note that neither (i) implies (ii) nor the converse. Even though the parametric optimizer in Robinson's counterexample is not locally Lipschitz, Proposition 4 shows that it enjoys other, slightly weaker, regularity properties. In particular, this result implies that u_4^*, the fourth component of the parametric optimizer of Robinson's counterexample, is continuous, cf. Figure 1.

6. Existence and Uniqueness of Solutions under Optimization-Based Controllers

Here, we leverage the regularity properties established in Section 5 to study existence and uniqueness of solutions for the closed-loop system under the optimization-based controller. The following result establishes existence of solutions.

Proposition 5. (Existence of solutions for the closed-loop system): *Suppose that f and g are twice*

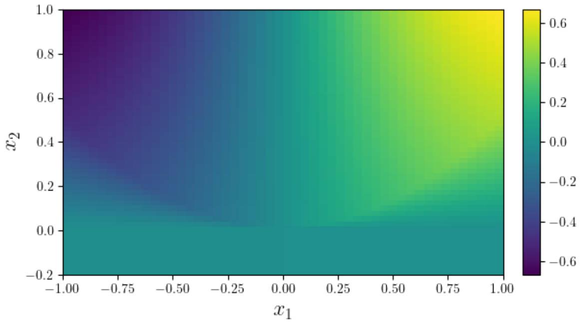

Figure 1: Numerical depiction of the fourth component of the parametric optimizer of Robinson's counterexample, cf. (1). The plot shows that it is continuous at the origin, in agreement with Proposition 4.

continuously differentiable in $\mathbb{R}^n \times \mathbb{R}^m$*. Further assume that given* $x_0 \in \mathbb{R}^n$*,* $f(\cdot, x_0)$ *is strongly convex,* $g(\cdot, x_0)$ *is convex and there exists* $\hat{u} \in \mathbb{R}^m$ *such that* $g(\hat{u}, x_0) < 0$*. Let* $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *be locally Lipschitz. Then, the differential equation*

$$\dot{x} = F(x, u^*(x))$$

with initial condition $x(0) = x_0$ *has at least one solution* $x : (-\delta, \delta) \to \mathbb{R}^n$ *for some* $\delta > 0$*.*

Proof. By Proposition 4, u^* has the Hölder property at x_0 and there exists a neighborhood $\mathcal{V}_{x_0}$ of x_0 such that u^* is point-Lipschitz at y for all $y \in \mathcal{V}_{x_0}$. Both of these properties imply that u^* is continuous in a neighborhood of x_0. The result follows by Peano's existence theorem [21, Theorem 2.1]. □

Next, we study uniqueness of solutions. The question we address is whether the assumptions of Proposition 5 are sufficient to ensure this property. We first note that the Hölder property does not imply uniqueness, even in simple one-dimensional examples. As an example, the differential equation $\dot{x} = x^{1/3}$ has the Hölder property at 0 but infinitely many solutions starting from the origin. The next example shows that, in general, point-Lipschitzness does not imply uniqueness of solutions either.

Example 6. (Point-Lipschitz differential equation with non-unique solutions): Let $u^* : \mathbb{R}^2 \to \mathbb{R}^4$ be the parametric optimizer of Robinson's counterexample. Consider the dynamical system

$$\dot{x}_1 = \frac{1}{2}, \tag{7a}$$

$$\dot{x}_2 = u_4^*(x_1, x_2), \tag{7b}$$

with initial condition $(x_1(0), x_2(0)) = (0, 0)$. Note that, by Proposition 4, the vector field in (7) is point-Lipschitz at the origin. Finally, note that (7) admits the following two solutions starting from the origin: $y_1(t) := (\frac{1}{2}t, 0)$ and $y_2(t) := (\frac{1}{2}t, \frac{1}{8}t^2)$, cf. Figure 2. •

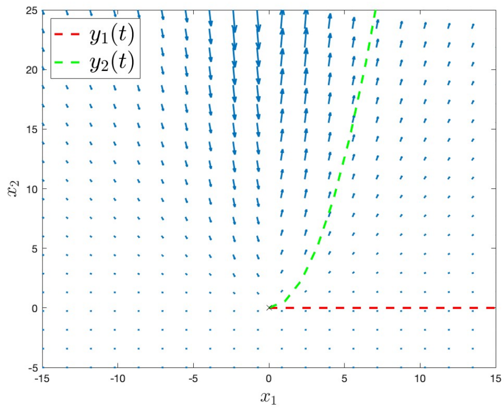

Figure 2: The blue arrows depict the vector field (7). The dashed red and green curves depict the two solutions y_1 and y_2 starting from the origin, where the vector field is point-Lipschitz but not locally Lipschitz.

Interestingly, the next result shows that point-Lipschitzness guarantees uniqueness of solutions starting from equilibria.

Proposition 7. (Point-Lipschitzness and Uniqueness): *Let* $\tilde{F} : \mathbb{R}^n \to \mathbb{R}^n$ *be point-Lipschitz at* $x_0 \in \mathbb{R}^n$ *and* $F(x_0) = \mathbf{0}_n$*. Then, there exists* $\delta > 0$ *such that the differential equation* $\dot{x} = \tilde{F}(x)$ *with initial condition* $x(0) = x_0$ *has only one solution, equal to* $x(t) = x_0$ *for* $t \in [0, \delta)$*.*

Proof. Let L be the point-Lipschitzness constant of $\tilde{F}$ and take $\delta < \frac{1}{L}$. Suppose that there exists another solution $y : [0, \delta) \to \mathbb{R}^n$ starting from x_0. Then, $\sup_{t \in [0,\delta)} \|y(t) - x_0\| > 0$. Moreover,

$$\sup_{t \in [0,\delta)} \|y(t) - x_0\| = \sup_{t \in [0,\delta)} \|\int_0^t \tilde{F}(y(s))ds\| =$$

$$\sup_{t \in [0,\delta)} \|\int_0^t \Big(\tilde{F}(y(s)) - \tilde{F}(x_0)\Big)ds\| \leq$$

$$\sup_{t \in [0,\delta)} \int_0^t L\|y(s) - x_0\|ds \leq L\delta \sup_{t \in [0,\delta)} \sup_{s \in [0,t]} \|y(s) - x_0\|$$

$$= L\delta \sup_{t \in [0,\delta]} \|y(t) - x_0\| < \sup_{t \in [0,\delta]} \|y(t) - x_0\|$$

where in the last inequality we have used the fact that $\sup_{[0,\delta)}\|y(t) - x_0\| > 0$. We hence reach a contradiction, which means that the constant solution is the only solution for $t \in [0, \delta)$. □

This result implies that in one dimension point-Lipschitz ODEs have unique solutions.

Corollary 8. (Point-Lipschitzness implies uniqueness in one dimension): *Let $\tilde{F} : \mathbb{R} \to \mathbb{R}$ be continuous in a neighborhood of x_0 and point-Lipschitz at x_0. Then, the differential equation $\dot{x} = \tilde{F}(x)$ with initial condition $x(0) = x_0$ has a unique solution.*

Proof. If $\tilde{F}(x_0) \neq 0$, by [22, Theorem 1.2.7], the differential equation has only one solution. If $\tilde{F}(x_0) = 0$, the result follows from Proposition 7. □

Since in general the assumptions of Proposition 5 are not sufficient to ensure uniqueness of solutions of the closed-loop system, additional assumptions must be made. Indeed, this has been explored in the literature [14] of optimization-based controllers. Under the additional assumption of *constant-rank constraint qualification*, the parametric solution u^* is locally Lipschitz [23, Theorem 3.6] and the closed-loop system has a unique solution. A similar result holds under the slightly stronger assumption that the gradients of the active constraints are linearly independent [5, Theorem 4.1]. Moreover, under the additional *strict complementary slackness* assumption, [24, Theorem 2.1] shows that u^* is continuously differentiable and, therefore, the closed-loop system also has unique solutions. This last point was already noted in [14, Theorem 1].

7. Conclusions

This note has sought to clarify the regularity properties enjoyed by parametric optimizers arising from optimization problems whose data satisfies the same hypotheses as Robinson's counterexample. We have shown that, even though the parametric optimizer in Robinson's counterexample is not locally Lispchitz, it enjoys other important regularity properties, like point-Lipschitzness. These are enough to guarantee existence of solutions of dynamical systems driven by optimization-based controllers but, in general, not uniqueness (for which otherwise stronger constraint qualifications must be satisfied), as we have illustrated with an example. We have identified cases where point-Lipschitzness is enough to guarantee uniqueness of solutions. The results presented in this note open the possibility of studying weaker conditions on the optimization problem that guarantee existence of solutions of the closed-loop system, possibly also using notions of solutions for discontinuous systems, like Carathéodory or Krasovskii solutions.

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