Regularity Properties of Optimization-Based Controllers

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Abstract

This paper studies regularity properties of optimization-based controllers, which are obtained by solving optimization problems where the parameter is the system state and the optimization variable is the input to the system. Under a wide range of assumptions on the optimization problem data, we provide an exhaustive collection of results about their regularity, and examine their implications on the existence and uniqueness of solutions and the forward invariance guarantees for the resulting closed-loop systems. We discuss the broad relevance of the results in different areas of systems and controls.

Keywords: Parametric optimization, optimization-based control, existence and uniqueness of solutions

1. Introduction

Optimization-based controllers are ubiquitous in numerous areas of systems and control including safety-critical control [1], model predictive control [2, 3], and online feedback optimization [4, 5]. Optimization-based controllers are a particular class of parametric optimization problems. The theory of parametric optimization [6, 7, 8] considers optimization problems that depend on a parameter and studies the regularity properties of the minimizers with respect to it. In optimization-based control, the parameter is the state and the optimization variable is the input.

Given a dynamical system (either in discrete or continuous time) with state $x \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$, an optimization-based controller is a feedback law obtained by solving a problem of the form

$$\underset{u\in\mathbb{R}^m}{\operatorname{argmin}}\ f(x,u) \tag{1a}$$

$$\text{s.t. } g(x,u) \leq 0 \tag{1b}$$

with $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$. Note that the system state x acts as a parameter in (1). Assuming that the optimizer of (1) is unique for every $x \in \mathbb{R}^n$, this defines a function $u^* : \mathbb{R}^n \to \mathbb{R}^m$, mapping each state to the optimizer of (1). This approach allows to encode desirable goals for controller synthesis both in the cost function f and in the constraints g. For instance, desirable performance objectives such as minimum control effort or maximizing convergence rate can be captured by the cost function, whereas the constraint functions can capture operational limitations on control effort and prescriptions to ensure properties such as closed-loop stability or safety. The flexibility of this synthesis approach makes it particularly attractive, but we should note the caveat that, in general, the controller u^* is not available in closed form. Instead, additional work needs to be performed in order to find the input by solving the resulting optimization problem (1). Independently of the computational aspects, one needs to ensure that the resulting controller behaves properly when employed to close the loop on the dynamical system, hence the importance of the study of the regularity properties of optimization-based controllers. Next we present different examples from the systems and controls literature where such controllers arise and motivate the importance of studying the regularity properties of u^*.

Example 1.1. (Control barrier and Lyapunov

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function-based control): In safety-critical applications, safe controllers are often designed through control barrier functions (CBF) [1]. Let $\dot{x} = F(x,u)$ be a control system with $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ locally Lipschitz. Assume that the set of safe states is given by the 0-superlevel set of a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$. This function is a CBF if, for every $x \in \mathbb{R}^n$, there exists $u \in \mathbb{R}^m$ such that $\nabla h(x)^T F(x,u) + \alpha(h(x)) \geq 0$, where $\alpha : \mathbb{R} \to \mathbb{R}$ is an extended class $\mathcal{K}$ function[1]. Any Lipschitz controller $u_{\text{cbf}} : \mathbb{R}^n \to \mathbb{R}^m$ that satisfies this inequality at every $x \in \mathbb{R}^n$ renders the closed-loop system safe (i.e., makes the 0-superlevel set of h forward invariant). This inequality can be incorporated as a constraint in an optimization problem defining a feedback controller. For example, given a nominal controller $k : \mathbb{R}^n \to \mathbb{R}^m$, designed with desirable properties such as asymptotic stability of an equilibrium point or minimizing a certain infinite-horizon optimal control cost, *safety filters* [9] seek to find the controller closest to k that satisfies the CBF constraint. Such controller can be found at every state $x \in \mathbb{R}^n$ by solving the following optimization problem:

$$u^*_{\text{sf}}(x) = \underset{u \in \mathbb{R}^m}{\operatorname{argmin}} \ \frac{1}{2} \|u - k(x)\|^2, \tag{2a}$$

$$\text{s.t. } \nabla h(x)^T F(x,u) + \alpha(h(x)) \geq 0, \tag{2b}$$

where again $\alpha : \mathbb{R} \to \mathbb{R}$ is an extended class $\mathcal{K}$ function. Often, one also seeks to endow safety filters with stability guarantees by employing control Lyapunov functions (CLF) [10]. Given a positive definite function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, V is a CLF if, for all $x \in \mathbb{R}^n \setminus \{0\}$, there exists $u \in \mathbb{R}^m$ such that $\nabla V(x)^T F(x,u) + W(x) \leq 0$, where $W : \mathbb{R}^n \to \mathbb{R}$ is a positive definite function. A locally Lipschitz controller $u_{\text{clf}} : \mathbb{R}^n \to \mathbb{R}^m$ that satisfies this inequality at every state $x \in \mathbb{R}^n$ renders the origin of the closed-loop system asymptotically stable. Given a CLF $V : \mathbb{R}^n \to \mathbb{R}$, one can seek to endow u_{sf} with stability guarantees by solving the following optimization problem at every $x \in \mathbb{R}^n$:

$$u^*_{\text{cc}}(x) = \underset{u \in \mathbb{R}^m}{\operatorname{argmin}} \ \frac{1}{2} \|u - k(x)\|^2, \tag{3a}$$

$$\text{s.t. } \nabla h(x)^T F(x,u) + \alpha(h(x)) \geq 0, \tag{3b}$$

$$\nabla V(x)^T F(x,u) + W(x) \leq 0, \tag{3c}$$

[1]A function $\alpha : \mathbb{R} \to \mathbb{R}$ is an extended class-$\mathcal{K}$ function if it is strictly increasing and $\alpha(0) = 0$.

Note that both (2) and (3) are special cases of (1). Similar optimization-based control designs of the form (1) leveraging CLFs and CBFs have been proposed in [11, 12, 13, 14, 15, 16, 17, 18], among many others. If the system is control-affine, as it is often the case in practice, then (2) and (3) are quadratic programs (QPs). Importantly, u^*_{sf} (resp. u^*_{cc}) is only guaranteed to be safe (resp. safe and stable) if it is locally Lipschitz. Hence, studying the regularity properties of (2) and (3) is critical to ensure that the closed-loop system has the desired safety and/or stability properties. Moreover, if u^* is locally Lipschitz, then the right-hand side of (18) is locally Lipschitz too, and then the Picard-Lindelöf theorem [19, Theorem 2.2] guarantees existence and uniqueness of solutions for small enough times. Similar regularity properties are also relevant in the study of the contraction properties of optimization-based controllers of the form (2) and (3), as shown in [20]. •

Example 1.2. (Online feedback optimization): Here we describe the problem of optimally regulating the steady-state output of a plant, a task often referred to as *online feedback optimization* [4, 5]. This problem arises in a variety of application areas including power systems [21, 22], network congestion control [23], and traffic networks [24]. In a typical set-up, the plant is modeled with the dynamics

$$\begin{aligned} \dot{x} &= F(x,u) \\ y &= G(x,u) \end{aligned} \tag{4}$$

where $G : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$ and $y \in \mathbb{R}^k$ denote the output. We assume that there exists a map $h : \mathbb{R}^m \to \mathbb{R}^k$, called the *steady-state map*, such that for each constant input $u \in \mathbb{R}^m$ and initial condition $x_0 \in \mathbb{R}^n$, the corresponding output of (4) satisfies $y(t) \to h(u)$ as $t \to \infty$. Consider the problem of driving the output to an optimal steady-state, formalized by the optimization

$$\min_{u \in \mathbb{R}^m, y \in \mathbb{R}^k} \quad \Phi(u,y) \tag{5a}$$

$$\text{s.t.} \quad (u,y) \in \mathcal{U} \times \mathcal{Y} \tag{5b}$$

$$y = h(u), \tag{5c}$$

where $\mathcal{Y} \subset \mathbb{R}^k$ denotes the set of valid outputs, $\mathcal{U} \subset \mathbb{R}^m$ denotes the set of valid inputs, and $\Phi : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$ denotes the cost of the corresponding input-output pair. The problem can equivalently be viewed as an optimization over a set of inputs alone by eliminating the variable y from (5):

$$\min_{u\in\mathbb{R}^m} \quad \Phi(u, h(u)) \tag{6a}$$
$$\text{s.t.} \quad (u, h(u)) \in \mathcal{U} \times \mathcal{Y}. \tag{6b}$$

Note that (5) and (6) are "static" problems, i.e., the state of the plant does not appear in the cost function or the constraints as a parameter. In practice, however, the steady-state map and the plant dynamics are only partially known, and subject to external disturbances or model uncertainties. This precludes one from directly solving either problem offline and simply applying the resulting input to (4) (this strategy is called *feedforward optimization*). Instead, one turns the "static" formulation into a "dynamic" one by solving the problem online and using real-time measurements of the output of the plant in place of a closed-form expression of the steady-state output. Formally, this amounts to replacing the expression y in (5), or $h(u)$ in (6), with $G(x, u)$:

$$u^*_{\text{ofo}}(x) = \underset{u\in\mathbb{R}^m}{\text{argmin}} \quad f(x, u) := \Phi(u, G(x, u)) \tag{7a}$$
$$\text{s.t.} \quad u \in \mathcal{U} \tag{7b}$$
$$G(x, u) \in \mathcal{Y}. \tag{7c}$$

The idea is that at each time instant, the output measurement is obtained online and fed back into (7), hence this strategy is called *online feedback optimization*. We note that (7) is of the form (1), by rewriting the input and output set inclusions as inequalities as in (1). In this setting, understanding the regularity properties of the closed-loop dynamics $\dot{x} = F(x, u^*_{\text{ofo}}(x))$ becomes necessary to ensure good performance of implementing (7) on a physical plant. Letting u_{ss} denote the solution to (6), one would be interested, for instance, in showing that $u^*_{\text{ofo}}(x(t)) \to u_{\text{ss}}$ and $y(t) \to h(u_{\text{ss}})$ as $t \to \infty$. We also note that if the cost function in (5) is time-varying, the resulting "dynamic" formulation (7) is also time-varying. However, the added time-dependence can be treated as an extra parameter in the optimization problem (7), and therefore the results outlined in this paper can also be applied in such time-dependent settings. •

Example 1.3. (Optimization algorithms as dynamical systems): Optimization algorithms can be viewed from the lens of dynamical systems [25, 26, 27]. In some cases, such dynamical systems are designed using ideas from optimization-based control. Here we discuss the *safe gradient flow* [28]. Consider a constrained optimization problem of the form

$$\min_{x\in\mathbb{R}^n} \quad \bar{f}(x), \tag{8a}$$
$$\text{s.t.} \quad \bar{g}(x) \leq 0 \tag{8b}$$

where $\bar{f} : \mathbb{R}^n \to \mathbb{R}$ and $\bar{g} : \mathbb{R}^n \to \mathbb{R}^m$ are continuously differentiable functions with Lipschitz continuous gradients, and $\frac{\partial \bar{g}}{\partial x}$ has full rank for all $x \in \mathbb{R}^n$. We consider the problem of designing a continuous-time dynamical system such that the feasible set $\mathcal{C} = \{x \in \mathbb{R}^n \mid \bar{g}(x) \leq 0\}$ is forward invariant and trajectories converge to solutions to (8). To solve this problem, we consider the integrator system,

$$\dot{x} = F(x, \xi) = \xi, \tag{9}$$

along with the feedback controller

$$\xi^*_\alpha(x) = \underset{\xi\in\mathbb{R}^n}{\text{argmin}} \ \frac{1}{2} \left\|\xi + \nabla \bar{f}(x)\right\|^2 \tag{10a}$$
$$\text{s.t} \quad \frac{\partial \bar{g}(x)}{\partial x}\xi \leq -\alpha \bar{g}(x), \tag{10b}$$

where $\alpha > 0$ is a design parameter. The closed-loop dynamics are referred to as the *safe gradient flow* (cf. [28, Section IV.A] for an alternative derivation of the safe gradient flow using techniques from the theory of control barrier functions). We note again that (10) is of the form (1). Establishing regularity properties of ξ^*_α such as continuity or local Lipschitzness is critical for the solutions of (10) to exist and be unique. These properties are then leveraged to study the convergence of the solutions of (10) to the optimizers of (8) while ensuring that the feasible set is forward invariant. •

Example 1.4. (Projected Dynamical Systems): Projected dynamical systems are a class of systems whose evolution is constrained to remain inside a subset $\mathcal{C} \subset \mathbb{R}^n$. They are are widely used for analyzing and solving nonlinear programs and variational inequalities [29] and have wide-ranging applications including network economics (e.g., for analyzing supply chain networks or financial markets) [30], power networks [21, 31], anti-windup controllers for feedback optimization [32], and traffic flows [33], to name a few. While projected dynamical systems have been considered in quite general settings, such as on Riemmanian manifolds [34], or with respect to oblique projections [35], here we restrict ourselves to the Euclidean case. In this case, projected dynamical systems typically take the form

$$\dot{x} = \Pi_{\mathcal{C}}[x, H(x)] \tag{11}$$

where $H : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field, and $v \mapsto \Pi_{\mathcal{C}}[x, v]$ is the projection onto the tangent cone of $\mathcal{C}$:

$$\Pi_{\mathcal{C}}[x, v] = \text{proj}_{T_{\mathcal{C}}(x)}(v).$$

Recently, projected dynamical systems have been reinterpreted from the viewpoint of control theory, to design anytime flows solving variational inequalities [36], and for understanding their relationship to controllers obtained using techniques from the theory of control barrier functions [37]. For example, (11) can be interpreted as the closed-loop dynamics corresponding to the system,

$$\dot{x} = H(x) + u \tag{12}$$

with the feedback controller

$$u^*_{\text{proj}}(x) = \underset{u \in \mathbb{R}^m}{\text{argmin}} \quad \|u\|^2 \tag{13a}$$

$$\text{s.t.} \qquad H(x) + u \in T_{\mathcal{C}}(x). \tag{13b}$$

Even though feedback controllers of the form (13) are discontinuous, the resulting closed-loop system may still be well behaved. In the case where $\mathcal{C}$ is convex, one can show existence and uniqueness [29, Theorem 2.5] of Carathéodory solutions (cf. [38] for notions of solutions to discontinuous systems), and forward invariance of the set $\mathcal{C}$ [39, Corollary 4.8]. With the additional assumption of strong monotonicity of H, asymptotic stability of the unique equilibrium (11) follows as well. The control-theoretic interpretation of projected dynamical systems highlights the complex relationship between the regularity properties of optimization-based feedback controllers and the dynamical properties of the resulting closed-loop system. In particular, it shows that from the perspective of control design, a feedback controller may achieve its intended objective (e.g., ensuring invariance of a safe set or stabilization to a desired equilibrium point) with relatively weak regularity properties. •

Example 1.5. (Model predictive control): Here we explain how (1) is also applicable to model predictive controllers. Consider a discrete-time dynamical system

$$x^+ = F(x, u), \tag{14}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. We consider the problem of optimally controlling (14) to minimize a running cost $\ell(x, u)$ while ensuring the state and input satisfy constraints $x \in \mathcal{X} \subset \mathbb{R}^n$ and $u \in \mathcal{U} \subset \mathbb{R}^m$. Model predictive control is a method for solving this problem by solving a finite-horizon optimal control problem and implementing its solution over (14) in a receding horizon fashion. Here we show that model predictive control schemes can be interpreted as a discrete-time analog of optimization-based feedback control discussed in previous examples. Let $N > 0$ be a time horizon, and $\mathbf{x} = (\hat{x}_0, \hat{x}_1, \dots, \hat{x}_N) \in \mathbb{R}^{n(N+1)}$ and $\mathbf{u} = (\hat{u}_0, \hat{u}_1, \dots, \hat{u}_{N-1}) \in \mathbb{R}^{mN}$ denote the state and input prediction sequences over the time horizon. Consider the following optimization problem

$$\mathbf{u}^*_{\text{mpc}}(x) = \underset{\mathbf{u},\mathbf{x}}{\text{argmin}}\, V_N(\hat{x}_N) + \sum_{k=0}^{N-1} \ell(\hat{x}_k, \hat{u}_k) \tag{15a}$$

$$\text{s.t.} \qquad \hat{x}_{k+1} = F(\hat{x}_k, \hat{u}_k) \tag{15b}$$

$$\hat{x}_k \in \mathcal{X},\ \hat{u}_k \in \mathcal{U} \tag{15c}$$

$$\hat{x}_N \in \mathcal{X}_f \tag{15d}$$

$$\hat{x}_0 = x \tag{15e}$$

$$k \in \{0, \dots, N-1\}, \tag{15f}$$

where $V_N : \mathbb{R}^n \to \mathbb{R}$ and $\mathcal{X}_f \subset \mathbb{R}^n$ denote the terminal cost and terminal constraint respectively (we refer the reader to [3] for conditions on these ingredients to ensure closed-loop stability.) Next, consider the augmented system

$$x^+ = \bar{F}(x, \mathbf{u}) \tag{16a}$$

$$= F(x, \hat{u}_0), \tag{16b}$$

which simply corresponds to implementing the first input in the sequence $\mathbf{u}$ to (14). Note that (15) is a parametric optimization problem, where the parameter corresponds to the current state x. Model predictive control studies the closed-loop system obtained by the dynamics (16) and the controller (15). As shown in [40], establishing the continuity properties of (15) is critical in proving stability and robustness properties of MPC-based controllers. •

As motivated by the examples provided above, studying the regularity properties of u^* is critical in order to establish different properties of interest for the closed-loop system, such as

(i) existence and uniqueness of solutions (for different notions of solution, such as classical, Carathéodory, and Filippov);

(ii) dynamical properties such as forward invariance of safe sets or stabilization to an equilibrium point;
(iii) convergence of optimization algorithms such as the safe gradient flow;
(iv) good performance of online feedback optimization-based controllers;
(v) stability and robustness properties of MPC based controllers.

Additionally, from a practical point of view, guaranteeing regularity properties for u^* such as continuity or local Lipschitzness is useful to ease the implementation of such controllers on digital platforms and avoid chattering behavior.

1.1. State of the Art

Having established the importance of characterizing the regularity properties of (1), we next discuss the state of the art. There are a variety of works in the literature [41, 42, 43, 44] and [3, Theorem 2.7] that use the theory of parametric optimization to guarantee local Lipschitz continuity or other regularity properties of optimization-based controllers. For example, the results in [41] give different conditions that ensure continuity and continuous differentiability of optimization-based controllers. However, they either require the rather strong assumption of *strict complementary slackness*, which is not satisfied in many cases of interest, or are limited to quadratic programs that satisfy a set of technical conditions. The paper [41] also revisits Robinson's counterexample, first introduced in [45], in the context of optimization-based control, which shows that even for optimization problems defined by well-behaved data (e.g., second-order continuously differentiable objective function and constraints, strongly convex objective function, and feasible set with non-empty interior, which are widely employed in the design of safe and stabilizing controllers, cf. Example 1.1), the resulting controller might not be locally Lipschitz. The result in [42, Theorem 3] is more general but only ensures continuity under *Slater's condition* and other regularity properties on the optimization problem. The regularity results in our previous work [43, 44] establish different Lipschitz continuity results for second-order convex programs, but are limited to this specific type of optimization problems. Finally, [3, Theorem 2.7] only guarantees continuity of optimization-based controllers derived from MPC. We also note that in some cases, u^* can be computed in closed-form, in which case the regularity properties of u^* can be evaluated directly without having to resort to the theory of parametric optimization. Examples of such explicitly computable optimization-based controllers are provided in [46, 12, 13, 47] and [3, Chapter 7]. We would also like to point out that even though this work is mostly focused on control laws obtained as the solution of optimization problems of the form (1), the regularity properties of other control designs has also been studied in the literature. For example, the celebrated Sontag's Universal Formula [48] provides a smooth control law for stabilization of control-affine systems. More recently, similar designs have been given in the context of safety-critical control [49, 43, 50, 51].

1.2. Paper Goals and Contributions

Our main goal in writing this paper is to provide an integrative presentation of insights and results about the regularity of optimization-based controllers. We present in Table 1 a comprehensive collection of results that offers the reader interested in using optimization-based controllers a one-stop destination to assess the regularity properties of their control design. The paper presents several results from the literature, but restated here for completeness from the perspective of optimization-based control. The paper also contains many novel results that help fill gaps in the state of the art.

In what follows, we assume that the control system operates in continuous time and is given by

$$\dot{x} = F(x, u), \tag{17}$$

where $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz. Hence, the closed-loop system takes the form

$$\dot{x} = F(x, u^*(x)). \tag{18}$$

On the technical level, the contributions of the paper are as follows. First, we show that under appropriate constraint qualifications and regularity properties of the optimization problem (1), the resulting optimization-based controller is continuous, locally Lipschitz, continuously differentiable, and even analytic. We provide specific conditions for each of these cases and observe that any of those conditions guarantee existence and uniqueness of solutions for the closed-loop system. Second, given the importance of Robinson's counterexample in showing that optimization-based controllers defined

<table>
<tr><th>Assumptions</th><th>Regularity of u^*</th><th>Existence</th><th>Uniqueness</th></tr>
<tr><td>f, g analytic
$f(x, \cdot)$ strictly convex $\forall x \in \mathbb{R}^n$
$g(x, \cdot)$ convex $\forall x \in \mathbb{R}^n$
Existence of minimizer
LICQ and SCS</td><td>analytic
cf. [52]</td><td>✓</td><td>✓</td></tr>
<tr><td>$f, g \in \mathcal{C}^p(\mathbb{R}^n \times \mathbb{R}^m)$
$p \in \mathbb{Z}_{>0}, p \geq 2$
$f(x, \cdot)$ strictly convex
$g(x, \cdot)$ convex
Existence of minimizer
LICQ and SCS</td><td>$\mathcal{C}^{p-1}$
cf. [52]</td><td>✓</td><td>✓</td></tr>
<tr><td>$f, g \in \mathcal{C}^2(\mathbb{R}^n \times \mathbb{R}^m)$
$f(x, \cdot)$ strictly convex $\forall x \in \mathbb{R}^n$
$g(x, \cdot)$ convex $\forall x \in \mathbb{R}^n$
Existence of minimizer, LICQ</td><td>Locally Lipschitz
cf. [53]</td><td>✓</td><td>✓</td></tr>
<tr><td>$f, g \in \mathcal{C}^2(\mathbb{R}^n \times \mathbb{R}^m)$
$f(x, \cdot)$ strictly convex $\forall x \in \mathbb{R}^n$
$g(x, \cdot)$ convex $\forall x \in \mathbb{R}^n$
Existence of minimizer
CR and MFCQ</td><td>Locally Lipschitz
cf. [54]</td><td>✓</td><td>✓</td></tr>
<tr><td>$f, g \in \mathcal{C}^2(\mathbb{R}^n \times \mathbb{R}^m)$
$f(x, \cdot)$ strongly convex $\forall x \in \mathbb{R}^n$
$g(x, \cdot)$ convex $\forall x \in \mathbb{R}^n$
Existence of minimizer, SC</td><td>Point-Lipschitz and Hölder,
cf. Proposition 3.2, and
locally Lipschitz for scalar
QPs, cf. Proposition 3.3</td><td>✓</td><td>Only in special cases
cf. Proposition 4.3
Corollary 4.4, Example 4.2</td></tr>
<tr><td>$f, g \in \mathcal{C}^2(\mathbb{R}^n \times \mathbb{R}^m)$
$f(x, \cdot)$ strictly convex $\forall x \in \mathbb{R}^n$
$g(x, \cdot)$ convex $\forall x \in \mathbb{R}^n$
Existence of minimizer, SC</td><td>Directionally differentiable,
cf. Proposition 3.2,
locally Lipschitz for scalar
QPs, cf. Proposition 3.3, and
continuous, cf. [55, Thm 5.3],
but might not be
point-Lipschitz,
cf. Example 3.4</td><td>✓</td><td>✗
cf. Example 4.2</td></tr>
<tr><td>$f, g \in \mathcal{C}^0(\mathbb{R}^n \times \mathbb{R}^m)$
$f(x, \cdot)$ strictly convex $\forall x \in \mathbb{R}^n$
$g(x, \cdot)$ convex $\forall x \in \mathbb{R}^n$
Existence of minimizer
LCF $\forall x \in \mathbb{R}^n$</td><td>Locally bounded
cf. Proposition 3.7, and
measurable
cf. Proposition 3.8</td><td>✗ (classical)
✓ (Filippov)</td><td>✗ (classical)
✗ (Filippov)</td></tr>
<tr><td>$f, g \in \mathcal{C}^2(\mathbb{R}^n \times \mathbb{R}^m)$
$f(x, \cdot)$ strongly convex $\forall x \in \mathbb{R}^n$
$g(x, \cdot)$ convex $\forall x \in \mathbb{R}^n$
Existence of minimizer</td><td>Might be discontinuous
and even unbounded
cf. Examples 3.5, 3.6</td><td>✗ (classical)
✗ (Filippov)</td><td>✗ (classical)
✗ (Filippov)</td></tr>
</table>

Table 1: Summary of results on regularity properties of optimization-based controllers. The first column describes the different assumptions. The second column describes the regularity properties of u^*. The third (resp. fourth) column describes whether existence (resp. uniqueness) of classical solutions of the closed-loop system (18) is guaranteed (provided that $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz). In the last two columns, properties are stated by default for classical solutions. If results are available for both classical and Filippov solutions, the property for each type of solution is denoted separately. LICQ stands for *linear independence constraint qualification*, SCS stands for *strict complementary slackness*, MFCQ stands for *Mangasarian-Fromovitz Constraint Qualification* and CR stands for *constant rank condition.* The terminology for regularity and constraint qualification is given in Section 2.

by well-behaved data might not be locally Lipschitz and its implications, e.g., for safety-critical control, cf. Example 1.1 (where most optimization-based controllers are defined by problem data sharing the properties of Robinson's counterexample), we characterize the regularity properties enjoyed by the parametric optimizer of problems defined by objective and constraints with the same properties as in Robinson's counterexample. We show that even though such parametric optimizers are not locally Lipschitz in general, they enjoy other weaker regularity properties, which are enough to guarantee existence of solutions for the closed-loop system and in some special cases, even uniqueness. Third, we provide different examples that show how if the properties in Robinson's counterexample do not hold, optimization-based controllers can be discontinuous and in some cases, even unbounded. Fourth, we show that even if the optimization-based controller is discontinuous, under appropriate regularity properties of the optimization problem (1), the parametric optimizer is measurable and locally bounded, and the closed-loop system has Filippov solutions. Finally, given a safe set of interest, we study under what regularity conditions on (1) and the set, solutions of the closed-loop system remain in the safe set, both for classical and Filippov solutions.

2. Preliminaries

In this section we discuss different preliminary results on regularity of functions and constraint qualifications.

2.1. Notions of regularity of functions

Throughout the note, we make use of the following notions of regularity of functions.

Definition 2.1. (Notions of Lipschitz continuity): *A function $f : \mathbb{R}^n \to \mathbb{R}^q$ is*

- point-Lipschitz at $x_0 \in \mathbb{R}^n$ *if there exists a neighborhood $\mathcal{U}$ of x_0 and a constant $L \geq 0$ such that*

$$\|f(x) - f(x_0)\| \leq L \|x - x_0\|, \quad \forall x \in \mathcal{U}. \tag{19}$$

- locally Lipschitz at $x_0 \in \mathbb{R}^n$ *if there exists a neighborhood $\tilde{\mathcal{U}}$ of x_0 and a constant $\tilde{L} \geq 0$ such that*

$$\|f(x) - f(y)\| \leq \tilde{L} \|x - y\|, \quad \forall x, y \in \tilde{\mathcal{U}}. \tag{20}$$

The notion of point-Lipschitz continuity is used, for instance, in [7, Section 6.3] and called *Lipschitz stability*, without clearly acknowledging the difference with the notion of local Lipschitz continuity. In the optimization literature, this property is sometimes referred to as *calmness* (cf. [56, Chapter 8.F]). Studying point-Lipschitz continuity is natural in the context of parametric optimization, as one is normally interested in understanding the changes in the solution with respect to a *fixed* value of the parameter. Locally Lipschitz functions are point-Lipschitz, but the converse is not true. For instance, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x \sin(\frac{1}{x})$ if $x \neq 0$ and $f(0) = 0$ is point-Lipschitz but not locally Lipschitz at the origin.

Definition 2.2. (Hölder property): *A function $f : \mathbb{R}^n \to \mathbb{R}^q$ has the* Hölder property at $x_0 \in \mathbb{R}^n$ *if there exists a neighborhood $\hat{\mathcal{U}}$ of x_0 and constants $C > 0$, $\alpha \in (0, 1]$ such that*

$$\|f(x) - f(y)\| \leq C \|x - y\|^{\alpha}, \quad \forall x, y \in \hat{\mathcal{U}}. \tag{21}$$

Note that if f is locally Lipschitz at x_0 then it also has the Hölder property at x_0 but the converse is not true.

Definition 2.3. (Directionally differentiable function): *A function $f : \mathbb{R}^n \to \mathbb{R}$ is directionally differentiable if for any vector $v \in \mathbb{R}^n$, the limit*

$$\lim_{h \to 0} \frac{f(x + hv) - f(x)}{h}$$

exists. A vector-valued function is directionally differentiable if each of its components is directionally differentiable.

Let $\Omega \subset \mathbb{R}^n$. Throughout the paper, a function $\varphi : \Omega \to \mathbb{R}^d$ belongs to the set $\mathcal{C}^k(\Omega)$ if φ is k-times continuously differentiable in Ω. A function $\varphi : \Omega \to \mathbb{R}^d$ belongs to the set $\mathcal{C}^0(\Omega)$ if φ is continuous in Ω. In case we view the elements in Ω as vectors of the Cartesian product $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ and φ takes the form $(x, u) \mapsto \varphi(x, u)$, the function $\varphi \in \mathcal{C}^{0,k}(\Omega)$ if for every $x \in \Omega$, the derivatives of order up to k of $\varphi(x, \cdot)$ with respect to u exist and are continuous with respect to x and u.

Definition 2.4. (Analytic function): *A function $f : \mathbb{R}^n \to \mathbb{R}$ is analytic in an open set D if for any $x \in D$ there exists a sequence $\{a_n\}_{n \in \mathbb{Z}_{\geq 0}}$ such that $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ for all x in a neighborhood of x_0. A vector-valued function is analytic in an open set D if each of its components is analytic.*

Note that an analytic function in an open set D belongs to $\mathcal{C}^k(D)$ for any $k \in \mathbb{Z}_{\geq 0}$. Finally, we introduce the last notion of regularity, which is weaker than all the ones presented above and only requires the function to be bounded in a neighborhood of a point.

Definition 2.5. (Locally bounded function): *A function $f : \mathbb{R}^n \to \mathbb{R}^q$ is locally bounded at $x_0 \in \mathbb{R}^n$ if there exists a neighborhood $\breve{\mathcal{U}}$ of x_0 and a constant $B > 0$ such that $\|f(x)\| \leq B$ for all $x \in \breve{\mathcal{U}}$.*

2.2. Constraint Qualifications and Conditions

Here we recall different constraint qualifications and conditions for problem (1) following [57, 7]. Throughout this section we fix $x \in \mathbb{R}^n$. Furthermore, given $u \in \mathbb{R}^m$, we let $\mathcal{I}(x, u)$ be the set of active constraints at (x, u), i.e., $\mathcal{I}(x, u) := \{i \in \{1, \dots p\} \mid g_i(x, u) = 0\}$. We consider the following:

MFCQ: Mangasarian-Fromovitz Constraint Qualification (MFCQ) holds at $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$ if there exists $z \in \mathbb{R}^m$ such that $\nabla_u g_i(x, u)z < 0$ for all $i \in \mathcal{I}(x, u)$;

LICQ: Linear Independence Constraint Qualification (LICQ) holds at $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$ if the vectors $\{\nabla_u g_i(x, u), i \in \mathcal{I}(x, u)\}$ are linearly independent;

CR: Constant Rank condition (CR) holds at $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$ if for any subset $L \subset \mathcal{I}(x, u)$ of active constraints, there exists a neighborhood $\mathcal{N}$ of (x, u) such that the family $\{\nabla_u g_i(x, u), i \in L\}$ remains of constant rank in $\mathcal{N}$;

SC: Slater's Condition (SC) holds at $x \in \mathbb{R}^n$ if there exists $\hat{u} \in \mathbb{R}^m$ such that $g_i(x, \hat{u}) < 0$ for all $i \in \{1, \dots, p\}$;

SCS: given $x \in \mathbb{R}^n$, let $(u^*(x), \lambda^*(x))$ be a KKT point for the optimization problem in (1). Then, $(u^*(x), \lambda^*(x))$ satisfies Strict Complementary Slackness (SCS) if there does not exist $i \in \{1, \dots, p\}$ such that $\lambda_i^*(x) = 0$ and $g_i(x, u^*(x)) = 0$;

LCF: local compact feasibility (LCF) holds at $x \in \mathbb{R}^n$ if there exists a compact set $K \subset \mathbb{R}^m$ and $\delta > 0$ such that for all $y \in \mathbb{R}^n$ such that $\|y - x\| < \delta$, there exists $u \in K$ such that $g(y, u) \leq 0$.

3. Regularity of Parametric Optimizers

In this section we discuss how the assumptions on the functions f and g defining (1) affect the regularity properties of the resulting controller u^*. Throughout this section, we assume that f and g belong to $\mathcal{C}^2(\mathbb{R}^n \times \mathbb{R}^m)$, $f(x, \cdot)$ is strictly convex for all $x \in \mathbb{R}^n$ (for some specific results, we further assume that $f(x, \cdot)$ is strongly convex for all $x \in \mathbb{R}^n$, but we make it explicit if this is the case) and $g(x, \cdot)$ is convex for all $x \in \mathbb{R}^n$. We further assume that, for each $x \in \mathbb{R}^n$, (1) has at least one minimizer (note that if $f(x, \cdot)$ is strongly convex for all $x \in \mathbb{R}^n$, this holds if (1) is feasible for all $x \in \mathbb{R}^n$). By the convexity assumptions described above, this implies that $u^*(x)$ is a singleton for every $x \in \mathbb{R}^n$. Furthermore, the assumptions on convexity also ensure that for each $x \in \mathbb{R}^n$, *interior-point algorithms* [58] can be used to solve (1), which run in polynomial time when the optimization problem is a linear program, a quadratic program, a second-order convex program, or a semidefinite program. This type of assumptions are very common, for instance, in CBF-based QPs, (cf. Example 1.1), or in model predictive controllers for linear systems (cf. Example 1.5), for which there also exist specific algorithms that solve them efficiently [59].

First, we gather a few existing results from the literature:

Continuity: Under the assumption that MFCQ holds at $(x, u^*(x))$, the parametric solution u^* is continuous [55, Theorem 5.3].

Local Lipschitzness: Under the assumption that both MFCQ and CR hold at $(x, u^*(x))$, the parametric solution u^* is locally Lipschitz [54, Theorem 3.6]. The same conclusion can be obtained if LICQ holds [53, Theorem 4.1] at $(x, u^*(x))$. We note also that since the satisfaction of LICQ implies the satisfaction of MFCQ and CR (cf. [54, Proposition 3.1]), [54, Theorem 3.6] is stronger than [53, Theorem 4.1].

Continuous Differentiability: Under the assumptions of LICQ and SCS, the parametric solution u^* is continuously differentiable [52, Theorem 2.1]. This last point was already noted in the optimization-based control literature in [41, Theorem 1]. In fact, if f and g belong to $\mathcal{C}^p(\mathbb{R}^n, \mathbb{R}^m)$, with $p \in \mathbb{Z}_{>0}, p \geq 2$, the proof of [52, Theorem 2.1] can be adapted using the Implicit Function Theorem for higher degree of differentiability [60, Proposition 1B.5], to show that the parametric optimizer belongs to $\mathcal{C}^{p-1}(\mathbb{R}^n)$.

Analyticity: Similarly, if f and g are analytic in $\mathbb{R}^n$, then the proof of [52, Theorem 2.1] can be adapted using the Analytic Function Theorem [61, Theorem 3.3.2], to show that the parametric optimizer is analytic in $\mathbb{R}^n$.

If the constraint qualifications given above for the case of local Lipschitzness do not hold, the parametric optimizer can fail to be locally Lipschitz. To illustrate this, we revisit next an example due to Robinson [45].

Example 3.1. (Robinson's Counterexample): In [45], Robinson introduces the following parametric optimization problem: for $x = (x_1, x_2) \in \mathbb{R}^2$, consider

$$\min_{u\in\mathbb{R}^4} \frac{1}{2}u^\top u \tag{22a}$$

$$\text{s.t. } A(x)u \geq b(x) \tag{22b}$$

where

$$A(x) = \begin{bmatrix} 0 & -1 & 1 & 0, \\ 0 & 1 & 1 & 0, \\ -1 & 0 & 1 & 0, \\ 1 & 0 & 1 & x_1 \end{bmatrix}, \quad b(x) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1+x_2 \end{bmatrix}.$$

Problem (22) is a quadratic program with strongly convex objective function, smooth objective function and constraints, and for which Slater's condition holds for every value of the parameter (this can be shown by noting that $\hat{u} = (0, 0, 2 + |x_2|, 0)$ satisfies all constraints strictly). Despite these nice properties, the parametric solution of (22) is not locally Lipschitz at $(x_1, x_2) = (0, 0)$. Indeed, let $u^* : \mathbb{R}^2 \to \mathbb{R}^4$ denote the parametric solution of (22), and $u_4^* : \mathbb{R}^2 \to \mathbb{R}$ its fourth component, which is given by

$$u_4^*(x_1, x_2) = \begin{cases} 0 & \text{if } x_2 \leq 0, \\ \frac{x_2}{x_1} & \text{if } x_2 \geq 0,\ x_1 \neq 0, \frac{x_1^2}{2} \geq x_2, \\ \frac{x_1(x_2+1)}{x_1^2+2} & \text{otherwise.} \end{cases}$$

u_4^* in Figure 1. For $s \geq 0$, let $p(s) = (s, s^2/2)$ and $q(s) = (s, 0)$. Then, observe that $p(s)$ and $q(s)$ approach the origin as $s \to 0^+$, however,

$$\frac{\|u_4^*(p(s)) - u_4^*(q(s))\|}{\|p(s) - q(s)\|} = \frac{1}{s}.$$

Since the right hand side of the previous expression can be made arbitrarily large by choose s sufficiently small, it follows that u^* is not locally Lipschitz at the origin. We also note that [54, Example 3.11] gives a similar example for a parametric quadratic program with a two-dimensional optimization variable, three-dimensional parameter, strongly convex objective function, smooth objective function and constraints, and for which Slater's condition holds for every value of the parameter and the corresponding parametric optimizer also fails to be locally Lipschitz. •

Even though Example 3.1 shows that the parametric optimizer of (22) is not locally Lipschitz, it actually satisfies a set of weaker regularity properties. The following result characterizes them, in the general setting of optimization problems satisfying the same conditions as (22).

Proposition 3.2. (Regularity Properties of Parametric Optimizer): *Suppose that f and g belong to $\mathcal{C}^2(\mathbb{R}^n \times \mathbb{R}^m)$. Further assume that for any $x \in \mathbb{R}^n$, $g(x, \cdot)$ is convex. Suppose that SC holds at $x_0 \in \mathbb{R}^n$. Then,*

(i) *if $f(x, \cdot)$ is strongly convex for all $x \in \mathbb{R}^n$, there exists a neighborhood $\mathcal{V}_{x_0}$ of x_0 such that u^* is point-Lipschitz at y for all $y \in \tilde{\mathcal{V}}_{x_0}$;*

(ii) *if $f(x, \cdot)$ is strongly convex for all $x \in \mathbb{R}^n$, u^* has the Hölder property at x_0;*

(iii) *if $f(x, \cdot)$ is strictly convex for all $x \in \mathbb{R}^n$ and (1) has at least one minimizer for all $x \in \mathbb{R}^n$, u^* is directionally differentiable at x_0.*

Proof. First we note that since in (i) and (ii), $f(x_0, \cdot)$ is strongly convex, and in (iii) $f(x_0, \cdot)$ is strictly convex and (1) has at least one minimizer at x_0, it follows that in all of (i), (ii), (iii), $u^*(x_0)$ is unique and well-defined for all $x_0 \in \mathbb{R}^n$.

To prove (i) we use [7, Theorem 6.4]. Since SC holds at x_0, by [62, Prop. 5.39], since $g(x_0, \cdot)$ is convex, MFCQ holds at $(x_0, u^*(x_0))$. Furthermore, since $f(x_0, \cdot)$ is strongly convex and $g(x_0, \cdot)$ is convex, the second-order condition SOC2 [7, Definition 6.1] holds (note that SOC2 is not guaranteed to hold if $f(x_0, \cdot)$ is only strictly convex). All of this, together with the twice continuous differentiability of f and g imply, by [7, Theorem 6.4], that u^* is point-Lipschitz at x_0. Now, since g is continuous, there exists a neighborhood $\mathcal{V}_{x_0}$ of x_0 such that SC holds for all $y \in \mathcal{V}_{x_0}$. By repeating the same argument, u^* is point-Lipschitz at y for all $y \in \mathcal{V}_{x_0}$.

Now let us prove (ii). We use [63, Theorem 2.1], which gives a sufficient condition for the solution of a variational inequality to have the Hölder property. Recall that given a map $F : \mathbb{R}^m \to \mathbb{R}^m$, and a constraint set $\bar{\mathcal{C}} \subset \mathbb{R}^m$, a variational inequality refers to the problem of finding $u^* \in \bar{\mathcal{C}}$ such that $(u-u^*)^T F(u^*) \geq 0$ for all $u \in \bar{\mathcal{C}}$. For every fixed $x \in \mathbb{R}^n$, by taking the map F to be the gradient of f with respect to u at x, and by taking $\bar{\mathcal{C}}$ to be the constraint set of (1) at x, a constrained optimization problem of the form (1) can be posed as a variational inequality, cf. [64]. Since f is twice continuously differentiable and $f(x_0, \cdot)$ is strongly convex, conditions (2.1) and (2.2) in [63, Theorem 2.1] hold. Note that condition (2.2) in [63, Theorem 2.1] is not guaranteed to hold if $f(x_0, \cdot)$ is only strictly convex. Moreover, since MFCQ holds at $(x_0, u^*(x_0))$ (because SC holds), by [65, Remark 3.6] the constraint set is pseudo-Lipschitzian [63, Definition 1.1]. All of this implies by [63, Theorem 2.1] that u^* has the Hölder property at x_0.

Finally, (iii) follows from the fact that SC implies MFCQ and [66, Theorem 1]. Note that in this case, the assumptions of [66, Theorem 1] are satisfied by only requiring that $f(x, \cdot)$ is strictly convex for all $x \in \mathbb{R}^n$, instead of strongly convex for all $x \in \mathbb{R}^n$. □

In Proposition 3.2, note that neither (i) implies (ii) nor the converse. Even though the parametric optimizer in Robinson's counterexample is not locally Lipschitz, Proposition 3.2 shows that it enjoys other, slightly weaker, regularity properties. In particular, this result implies that u_4^*, the fourth component of the parametric optimizer of Robinson's counterexample, is continuous, cf. Figure 1.

Proposition 3.2 also clarifies a confusion that has arisen in the literature due to the loose use of terminology. Indeed, according to [7, Theorem 6.4], a parametric optimization problem whose data satisfies the properties of Robinson's counterexample has a Lipschitz minimizer! This apparent contradiction is rooted in different notions of Lipschitzness. Indeed, the notion of Lipschitzness used in [7, Theorem 6.4] corresponds to point-Lipschitzness.

Next we show that in the special case of parametric quadratic programs that satisfy the assumptions of Proposition 3.2 with a scalar optimization variable, the parametric optimizer is locally Lipschitz.

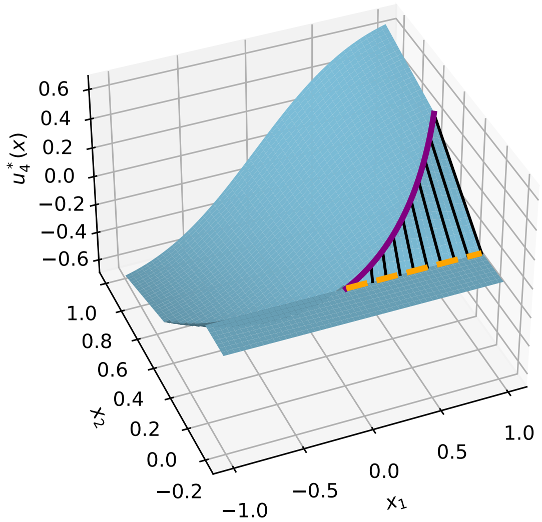

Figure 1: Surface plot of u_4, which is the fourth component of the parametric optimizer of Robinson's counterexample, cf. (22). The purple solid and orange dashed lines correspond to the plot of $u_4(p(s))$ and $u_4(q(s))$ (defined in Example 3.1) respectively, for $s \geq 0$. The solid black lines connect points corresponding to $p(s)$ and $q(s)$. The plot shows that u_4 is continuous at the origin, in agreement with Proposition 3.2. However, since the slope of the line connecting $u_4(p(s))$ and $u_4(q(s))$ becomes arbitrarily large as p and q approach the origin, u_4 is not locally Lipschitz at the origin.

Proposition 3.3. (Scalar parametric quadratic programs have locally Lipschitz optimizers): *Suppose that $f \in \mathcal{C}^2(\mathbb{R}^n \times \mathbb{R}^1)$ and and for $i \in \{1, \ldots, p\}$, let $g_i^0 : \mathbb{R}^n \to \mathbb{R}$, $g_i^1 : \mathbb{R}^n \to \mathbb{R}$ belong to $\mathcal{C}^2(\mathbb{R}^n)$ and*

$$g(x,u) = \begin{pmatrix} g_1^0(x)u + g_1^1(x) \\ \vdots \\ g_i^0(x)u + g_i^1(x) \\ \vdots \\ g_p^0(x)u + g_p^1(x) \end{pmatrix}.$$

Further assume that for any $x \in \mathbb{R}^n$, $f(x, \cdot)$ is strictly convex and (1) has at least one minimizer. Suppose that SC holds at $x_0 \in \mathbb{R}^n$. Then, u^* is locally Lipschitz at x_0.

Proof. First, since (1) has at least one minimizer for all $x \in \mathbb{R}^n$, the convexity assumptions of f and g imply that u^* is a singleton for all $x \in \mathbb{R}^n$. Note that for all $i \in \mathcal{I}(x_0, u^*(x_0))$, $g_i^0(x_0) \neq 0$. Indeed, if $g_i^0(x_0) = 0$ and $i \in \mathcal{I}(x_0, u^*(x_0))$, it follows that $g_i^1(x_0) = 0$, which implies that Slater's condition at x_0 is violated. Hence, $g_i^0(x_0) \neq 0$ for all $i \in \mathcal{I}(x_0, u^*(x_0))$ and the CR holds at $(x_0, u^*(x_0))$. Moreover, since Slater's condition holds at x_0, by [62, Prop. 5.39], since $g(x_0, \cdot)$ is convex, MFCQ holds at $(x_0, u^*(x_0))$. By [54, Theorem 3.6], this implies that u^* is locally Lipschitz at x_0. □

Note that Robinson's counterexample or [54, Example 3.11] do not contradict Proposition 3.3, since in those two examples the optimization variable of the quadratic program has dimensions four and two, respectively. Proposition 3.3 shows that optimization-based controllers for single-input systems with affine constraints (e.g., obtained from CBF or CLF based conditions for control-affine systems) that satisfy Slater's conditions are locally Lipschitz.

The following examples show that the results from Proposition 3.2 do not hold if the assumptions are weakened, even slightly.

Example 3.4. (Not point-Lipschitz optimizer without differentiability of problem data with respect to the parameter): If f and g are not differentiable with respect to the parameter x but the rest of the assumptions of Proposition 3.2 hold (even with $f(x, \cdot)$ strongly convex for all $x \in \mathbb{R}^n$), the following example, inspired by Robinson's counterexample, shows that the parametric optimizer is not necessarily point-Lipschitz. Let $x = (x_1, x_2) \in \mathbb{R}^2$ and consider (22) with

$$A(x) = \begin{bmatrix} 0 & -1 & 1 & 0, \\ 0 & 1 & 1 & 0, \\ -1 & 0 & 1 & 0, \\ 1 & 0 & 1 & \sqrt{|x_1|} \end{bmatrix}, \quad b(x) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 + x_2 \end{bmatrix}.$$

Let $\tilde{u}^* : \mathbb{R}^2 \to \mathbb{R}^4$ be its parametric solution and let $\tilde{u}_4^* : \mathbb{R}^2 \to \mathbb{R}$ denote its fourth component, which is given by

$$\tilde{u}_4^*(x) = \begin{cases} 0 & \text{if } x_2 \leq 0, \\ \frac{x_2}{\sqrt{|x_1|}} & \text{if } x_2 \geq 0,\ x_1 \neq 0, \frac{|x_1|}{2} \geq x_2, \\ \frac{\sqrt{|x_1|}(x_2+1)}{|x_1|+2} & \text{otherwise.} \end{cases}$$

Let $x_1 > 0$ and define $p_{x_1} = (x_1, \frac{|x_1|}{2})$. Note that

$$\frac{\|\tilde{u}_4^*(p_{x_1}) - \tilde{u}_4^*(0)\|}{\|p_{x_1} - 0\|} = \frac{1}{\sqrt{5|x_1|}}.$$

Since x_1 can be taken to be arbitrarily small, $\tilde{u}^*$ is not point-Lipschitz at the origin. However, because f and g, as well as their first and second derivatives with respect to u, are continuous in u and x, and the rest of assumptions of Proposition 3.2 hold (with $f(x, \cdot)$ strongly convex for all $x \in \mathbb{R}^n$), then by [55, Theorem 5.3], the corresponding parametric optimizer, and hence $\tilde{u}_4^*$, is continuous. •

Example 3.5. (Discontinuous optimizer without Slater's condition): The following example, taken from [41, Section VI], shows that if Slater's condition does not hold, then continuity of the parametric optimizer is not guaranteed even if the rest of assumptions from Proposition 3.2 (even with $f(x, \cdot)$ strongly convex for all $x \in \mathbb{R}^n$) do hold:

$$\hat{u}^*(x) = \underset{u \in \mathbb{R}}{\operatorname{argmin}} \frac{1}{2} u^2 - 2u, \tag{23a}$$

$$\text{s.t.} \quad xu \leq 0. \tag{23b}$$

Indeed, the objective function and constraint of (23) are twice continuously differentiable, the objective function is strongly convex and the constraint is convex for any $x \in \mathbb{R}$. However, Slater's condition does not hold at $x = 0$. In fact,

$$\hat{u}^*(x) = \begin{cases} 2 & \text{if} \quad x \leq 0, \\ 0 & \text{else,} \end{cases}$$

is discontinuous at $x = 0$. However, note that $\hat{u}^*$ is bounded. •

Example 3.6. (Unbounded optimizer without Slater's condition): The following example, adapted from [47, Example III.5], shows that if Slater's condition fails, not only can the parametric optimizer fail to be continuous, as shown in Example 3.5, but it can even fail to be locally bounded. Let $x = (x_1, x_2) \in \mathbb{R}^2$, $a(x) = 2x_1 x_2 + x_2^2(1 - x_1^2 - x_2^2)$, and consider:

$$\breve{u}^*(x) = \underset{u \in \mathbb{R}}{\operatorname{argmin}} \frac{1}{2} \|u\|^2, \tag{24a}$$

$$\text{s.t.} \quad a(x) + 2x_2^3 u \leq 0. \tag{24b}$$

Note that Slater's condition does not hold at the point $x = (1, 0)$. Moreover, $\breve{u}^*$ is given by:

$$\breve{u}^*(x) = \begin{cases} 0 & \text{if} \quad a(x) \leq 0, \\ -\frac{a(x)}{2x_2^3} & \text{else.} \end{cases}$$

Note that $a(1, 0) = 0$ and $a(1, \epsilon) = 2\epsilon - \epsilon^4$. Moreover, since any neighborhood of $(1, 0)$ contains points of the form $(1, \epsilon)$ for sufficiently small $\epsilon > 0$, for any neighborhood $\mathcal{N}$ of $(1, 0)$ there exists $\epsilon_{\mathcal{N}} > 0$ sufficiently small such that $a(1, \epsilon_{\mathcal{N}}) > 0$. Now, since

$$\lim_{\epsilon \to 0} \frac{a(1, \epsilon)}{2\epsilon^3} = \infty,$$

and $\breve{u}^*(x) = -\frac{a(x)}{2x_2^3}$ if $a(x) > 0$, it follows that $\breve{u}^*$ is not locally bounded. •

Discontinuous controllers are relevant, and even necessary, in multiple applications, cf. [38]. When dealing with discontinuous systems, one needs to ensure basic properties such as local boundedness and measurability. In the following, we provide conditions that guarantee these properties for optimization-based controllers.

The following result gives a condition which ensures that parametric optimizers are locally bounded, hence precluding the behavior exhibited in Example 3.6.

Proposition 3.7. (Conditions for local boundedness): *Suppose that f and g belong to $\mathcal{C}^0(\mathbb{R}^n \times \mathbb{R}^m)$. Further assume that for any $x \in \mathbb{R}^n$, $f(x, \cdot)$ is strictly convex, $g(x, \cdot)$ is convex, and* (1) *has at least one minimizer. Then, given $x_0 \in \mathbb{R}^n$, u^* is locally bounded at x_0 if and only if LCF holds at x_0.*

Proof. Note that since (1) has at least one minimizer for all $x \in \mathbb{R}^n$, the convexity assumptions on f and g imply that u^* is a singleton for all $x \in \mathbb{R}^n$. First suppose that LCF holds at x_0. Therefore, there exists a compact set $K \subset \mathbb{R}^m$ and $\delta > 0$ such that for all $y \in \mathbb{R}^n$ such that $\|y - x\| < \delta$, there exists $u \in K$ such that $g(y, u) \leq 0$. Since f is continuous and K is compact, there exists $B_f > 0$ such that $|f(y, u)| < B_f$ for all $u \in K$ and $y \in \mathbb{R}^n$ such that $\|y - x_0\| < \delta$. Since for all $y \in \mathbb{R}^n$ such that $\|y - x_0\| < \delta$, there exists a feasible $u \in K$, it follows that $|f(y, u^*(y))| < B_f$ for all $y \in \mathbb{R}^n$ such that $\|y - x_0\| < \delta$. This implies that u^* is locally bounded at x_0. Now suppose that u^* is locally bounded at x_0 and suppose, by contradiction, that LFC does not hold at x_0. Then, for any $\delta > 0$ and compact set K, there exists $y \in \mathbb{R}^n$ with $\|y - x\| \leq \delta$ and such that all $u \in \mathbb{R}^m$ with $g(y, u) \leq 0$ satisfy $u \notin K$. This means that there exists a sequence $\{y_n\}_{n \in \mathbb{Z}_{>0}}$ such that $\|y_n - x\| \leq 1/n$ and $\|u^*(y_n)\| \geq n$ for all $n \in \mathbb{Z}_{>0}$, which implies that u^* is not locally bounded, hence reaching a contradiction. □

Verifying the local compact feasibility property can be challenging in general. However, for the particular case of CBF-based quadratic programs, [47, Theorem V.1] gives an alternative sufficient condition for local boundedness of u^* that only requires solving a specific linear equation.

Next, we turn our attention to the measurability properties of u^*.

Proposition 3.8. (Sufficient conditions for measurability): *Suppose that f and g belong to $\mathcal{C}^0(\mathbb{R}^n \times \mathbb{R}^m)$. Further assume that for any $x \in \mathbb{R}^n$, $f(x, \cdot)$ is strictly convex, $g(x, \cdot)$ is convex, and* (1) *has at least one minimizer. Further assume that for every $x \in \mathbb{R}^n$, LCF holds at x. Then, u^* is measurable.*

Proof. Note that since (1) has at least one minimizer for all $x \in \mathbb{R}^n$, the convexity assumptions on f and g imply that u^* is a singleton for all $x \in \mathbb{R}^n$. We use the Measurable Maximum Theorem [67, Theorem 18.19]. We assume that $\mathbb{R}^n$ and $\mathbb{R}^m$ are equipped with the usual Borel σ-algebras. Since f is continuous, it is a Carathéodory function (cf. [67, Definition 4.50]). Therefore, we only need to ensure that the set-valued map $\phi : x \to \{u \in \mathbb{R}^m \mid g(x, u) \leq 0\}$ is a weakly measurable correspondence (cf. [67, Definition 18.1]) with nonempty compact values. The fact that ϕ takes nonempty values follows from the fact that the feasible set $\{u \in \mathbb{R}^m \mid g(x, u) \leq 0\}$ is nonempty. Moreover, since Proposition 3.7 ensures that u^* is locally bounded at every $x \in \mathbb{R}^n$, without loss of generality we can assume that ϕ takes compact values (otherwise, we can define extra constraints that ensure that the feasible set is bounded for every $x \in \mathbb{R}^n$ without changing the optimizer u^*). Now, to show that ϕ is a weakly measurable correspondence, we follow an argument similar to the proof of [67, Corollary 18.8]. For every $n \in \mathbb{Z}_{>0}$, define the set-valued map $\phi_n : x \to \{u \in \mathbb{R}^m \mid g(x, u) \leq 1/n\}$. By Lemma [67, Corollary 18.8], ϕ_n is measurable. Moreover, for every $x \in \mathbb{R}^n$ and $n \in \mathbb{Z}_{>0}$, $\phi(x) \subset \partial(\phi_n(x))$ (where $\partial(\phi_n(x))$ denotes the boundary of $\phi_n(x)$), and $\phi(x) = \cap_{n=1}^{\infty} \partial(\phi_n(x))$. Furthermore, again without loss of generality, $\partial(\phi_n)$ has compact values for every $n \in \mathbb{Z}_{>0}$ (again, if that is not the case we can define extra constraints that ensure that this holds without changing the optimizer u^*), and by [67, Theorem 18.4(3)], the intersection $\phi : x \to \cap_{n=1}^{\infty} \partial(\phi_n(x))$ is measurable. □

The second column in Table 1 summarizes the different results discussed in this section.

Remark 3.9. (Verifying constraint qualifications and conditions in practice without knowledge of the optimizer): To show that u^* is locally Lipschitz at a point $x \in \mathbb{R}^n$ using [54, Theorem 3.6], we need to verify that both MFCQ and CR hold at $(x, u^*(x))$. Similarly, [53, Theorem 4.1] (resp. [52, Theorem 2.1]) require the verification of LICQ (resp. LICQ and SCS) at $(x, u^*(x))$. These results require knowledge of $u^*(x)$ to verify the corresponding property holds at x. However, in several applications it can be useful to know the regions where the controller u^* is discontinuous (for instance, to design safety-critical controllers that avoid such regions). Slater's condition is useful for this purpose because Proposition 3.2 guarantees different regularity properties of u^* at x without requiring knowledge of $u^*(x)$ (assuming that the extra conditions on differentiability and convexity of the objective function and constraints in Proposition 3.2 also hold). Moreover, in the special case where the constraints in (1) are affine, i.e.,

$$g(x,u) = \begin{pmatrix} g_1^0(x)^T u + g_1^1(x) \\ \vdots \\ g_i^0(x)^T u + g_i^1(x) \\ \vdots \\ g_p^0(x)^T u + g_p^1(x) \end{pmatrix},$$

for $i \in \{1,\dots,p\}$ and $g_i^0 : \mathbb{R}^n \to \mathbb{R}^m$ and $g_i^1 : \mathbb{R}^n \to \mathbb{R}$ in $\mathcal{C}^2(\mathbb{R}^n)$, then [68] shows that by letting c_x^* be the optimal value of the linear program

$$\max_{u\in\mathbb{R}^m} \sum_{i=1}^m |u_i| \tag{25a}$$

$$\text{s.t.} \quad u_i \geq 0,\ i \in \{1,\dots,m\}, \tag{25b}$$

$$\sum_{i=1}^m u_i g_i^0(x) = 0, \tag{25c}$$

$$\sum_{i=1}^m u_i g_i^1(x) = 0. \tag{25d}$$

then Slater's condition holds at x if and only if $c_x^* = 0$. Hence, (25) can be solved before solving (1) to verify that u^* satisfies the regularity properties in Proposition 3.2. •

4. Existence and Uniqueness of Solutions under Optimization-Based Controllers

In this section, we leverage the regularity properties established in Section 3 to study existence and uniqueness of solutions for the closed-loop system (18) under the optimization-based controller u^*.

First, we note that by the Picard-Lindelöf theorem [19, Theorem 2.2], any of the assumptions described in Section 3 that guarantee that u^* is locally Lipschitz at a point x_0 also guarantee that the closed-loop system (18) has a unique solution with initial condition at x_0 for sufficiently small times.

The following result establishes existence of solutions under weaker assumptions.

Proposition 4.1. (Existence of classical solutions for the closed-loop system): *Suppose that f and g belong to $\mathcal{C}^{0,2}(\mathbb{R}^n \times \mathbb{R}^m)$. Further assume that for any $x \in \mathbb{R}^n$, $f(x,\cdot)$ is strictly convex, $g(x,\cdot)$ is convex, and (1) has at least one minimizer. Further assume that SC holds at $x_0 \in \mathbb{R}^n$. Let $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be locally Lipschitz. Then, there exists $\delta_0 > 0$ such that the differential equation (18) has at least one solution $x : (-\delta_0, \delta_0) \to \mathbb{R}^n$ with initial condition $x(0) = x_0$.*

Proof. Note that since (1) has at least one minimizer for all $x \in \mathbb{R}^n$, the convexity assumptions on f and g imply that u^* is a singleton for all $x \in \mathbb{R}^n$. Since SC holds at x_0, by [62, Prop. 5.39], since $g(x_0,\cdot)$ is convex, MFCQ holds at $(x_0, u^*(x_0))$. By [55, Theorem 5.3], this implies that u^* is continuous at x_0. Since g is continuous, there exists a neighborhood $\mathcal{V}_{x_0}$ of x_0 such that SC holds for all $y \in \mathcal{V}_{x_0}$. By the same argument, this implies that u^* is continuous in $\mathcal{V}_{x_0}$. The result now follows by Peano's existence theorem [69, Theorem 2.1]. □

Proposition 4.1 implies in particular that, under the assumptions of Proposition 3.2, the closed-loop system (18) has at least one solution in a neighborhood of x_0.

Next, we study uniqueness of solutions under the assumptions of Proposition 3.2. We first note that the Hölder property does not imply uniqueness, even in simple one-dimensional examples. For example, the differential equation $\dot{x} = x^{1/3}$ has the Hölder property at 0 but infinitely many solutions starting from the origin. The next example shows that, in general, point-Lipschitz continuity does not imply uniqueness of solutions either.

Example 4.2. (Point-Lipschitz differential equation with non-unique solutions): Let $u^* : \mathbb{R}^2 \to \mathbb{R}^4$ be the parametric optimizer of Robinson's counterexample. Consider the dynamical system

$$\dot{x}_1 = \frac{1}{2}, \tag{26a}$$

$$\dot{x}_2 = u_4^*(x_1, x_2), \tag{26b}$$

with initial condition $(x_1(0), x_2(0)) = (0,0)$. By Proposition 3.2, the vector field in (26) is point-Lipschitz at the origin. However, (26) admits the following two distinct solutions starting from the origin: $y_1(t) := (\frac{1}{2}t, 0)$ and $y_2(t) := (\frac{1}{2}t, \frac{1}{8}t^2)$, cf. Figure 2. •

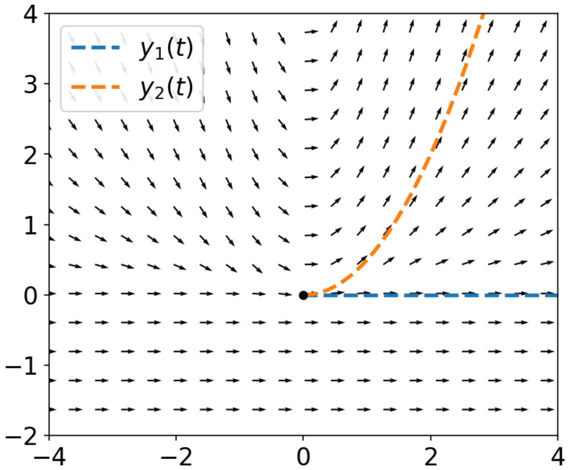

Figure 2: The arrows depict the vector field (26). The dashed blue and orange curves depict the two solutions y_1 and y_2 starting from the origin, where the vector field is point-Lipschitz but not locally Lipschitz.

Hence, in general the assumptions of Proposition 3.2 are not sufficient to ensure uniqueness of solutions of the closed-loop system. Interestingly, the next result shows that point-Lipschitz continuity guarantees uniqueness of solutions starting from equilibria.

Proposition 4.3. (Point-Lipschitz continuity and uniqueness): *Let $\tilde{F} : \mathbb{R}^n \to \mathbb{R}^n$ be point-Lipschitz at $x_0 \in \mathbb{R}^n$ and $\tilde{F}(x_0) = 0$. Then the function $x(t) = x_0$ for all $t \geq 0$ is the unique solution to the differential equation $\dot{x} = \tilde{F}(x)$ with initial condition $x(0) = x_0$.*

Proof. Let $\delta > 0$ and L be the point-Lipschitz continuity constant of $\tilde{F}$ and take $\delta < \frac{1}{L}$. Suppose that there exists another solution $y : [0, \delta) \to \mathbb{R}^n$ starting from x_0. Then, $\sup_{t\in[0,\delta)} \|y(t) - x_0\| > 0$. Moreover,

$$\sup_{t\in[0,\delta)} \|y(t) - x_0\| = \sup_{t\in[0,\delta)} \left\| \int_0^t \tilde{F}(y(s))ds \right\| =$$
$$\sup_{t\in[0,\delta)} \left\| \int_0^t \Big(\tilde{F}(y(s)) - \tilde{F}(x_0)\Big)ds \right\| \leq$$
$$\sup_{t\in[0,\delta)} \int_0^t L\,\|y(s) - x_0\|\, ds \leq$$
$$L\delta \sup_{t\in[0,\delta]} \|y(t) - x_0\| < \sup_{t\in[0,\delta]} \|y(t) - x_0\|$$

where in the last inequality we have used the fact that $\sup_{t\in[0,\delta)} \|y(t) - x_0\| > 0$. We hence reach a contradiction, which means that the constant solution is the only solution for $t \in [0, \delta)$. By repeating the same argument at time δ, we can extend this constant solution for all positive times. □

This result implies that in one dimension, point-Lipschitz ODEs have unique solutions.

Corollary 4.4. (Point-Lipschitz continuity implies uniqueness in one dimension): *Let $\tilde{F} : \mathbb{R} \to \mathbb{R}$ be continuous in a neighborhood of x_0 and point-Lipschitz at x_0. Then, the differential equation $\dot{x} = \tilde{F}(x)$ with initial condition $x(0) = x_0$ has a unique solution.*

Proof. If $\tilde{F}(x_0) \neq 0$, by [70, Theorem 1.2.7], the differential equation has only one solution. If $\tilde{F}(x_0) = 0$, the result follows from Proposition 4.3. □

If Slater's condition does not hold but the rest of assumptions of Proposition 3.2 hold (even with $f(x, \cdot)$ strongly convex for all $x \in \mathbb{R}^n$), Example 3.5 shows that u^* can be discontinuous, in which case neither existence nor uniqueness of solutions is guaranteed. In the case where f and g are not differentiable with respect to the parameter, but the rest of assumptions of Proposition 3.2 hold (even with $f(x, \cdot)$ strongly convex for all $x \in \mathbb{R}^n$), Example 3.4 shows that u^* is continuous but not necessarily point-Lipschitz. Therefore, in this case existence is guaranteed but uniqueness is not.

Note that so far, we have only considered existence and uniqueness for classical solutions. For discontinuous dynamical systems, other notions of solution, such as Carathéodory or Filippov solutions, can be defined, cf. [38]. The following result gives conditions on (1) that ensure the existence of Filippov solutions for (18).

Proposition 4.5. (Existence of Filippov solutions for the closed-loop system): *Suppose that f and g belong to $\mathcal{C}^2(\mathbb{R}^n \times \mathbb{R}^m)$. Further assume that for any $x \in \mathbb{R}^n$, $f(x,\cdot)$ is strictly convex, $g(x,\cdot)$ is convex and* (1) *has at least one minimizer. Finally suppose that for every $x \in \mathbb{R}^n$, LCF holds at x and $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz. Then for any $x \in \mathbb{R}^n$, there exists $\delta_x > 0$ such that* (18) *has at least one Filippov solution $y : (-\delta_x, \delta_x) \to \mathbb{R}^n$ with initial condition $y(0) = x$.*

Proof. Note that since (1) has at least one minimizer for all $x \in \mathbb{R}^n$, the convexity assumptions on f and g imply that u^* is a singleton for all $x \in \mathbb{R}^n$. By Propositions 3.7 and 3.8, the assumptions ensure that u^* is measurable and locally bounded. The result follows from [71, Theorem 7]. □

A weaker condition to ensure that Filippov solutions are unique is that the closed-loop system (18) is *essentially one-sided Lipschitz* [38]. Although it is known how to verify this property for projected dynamical systems (see e.g., [29, proof of Theorem 2.7] for the Euclidean case, and [34, Proposition 6.12] for the Non-Euclidean case), to the best of our knowledge there exist no results in the parametric optimization literature that guarantee that the parametric optimizer u^*, or the closed-loop dynamics, satisfies this property.

Finally, we note that if f is not strictly convex or g is not convex, the optimizer u^* is not guaranteed to be single-valued, which means that the usual notions of regularity of the controller and of solutions of the closed-loop system are not well defined. The third and fourth columns of Table 1 summarize the results presented in this section.

5. Forward Invariance Properties of Optimization-Based Controllers

In this section we study conditions that guarantee the forward invariance of a set for the closed-loop system under an optimization-based controller. Recall the notion of *tangent cone* to a set $\mathcal{C} \subset \mathbb{R}^n$:

The basic result concerning forward invariance is the following:

Theorem 5.1. (Nagumo's Theorem [72, 73]): *Let $\tilde{F} : \mathbb{R}^n \to \mathbb{R}^n$ and consider the system $\dot{x} = \tilde{F}(x)$. Assume that, for each initial condition in a set $\mathcal{D} \subset \mathbb{R}^n$, it admits a unique forward complete solution (i.e., a unique solution defined for all positive times). Let $\mathcal{C} \subset \mathcal{D} \subset \mathbb{R}^n$ be a closed set. Then the set $\mathcal{C}$ is forward invariant for the system if and only if $\tilde{F}(x) \in T_{\mathcal{C}}(x)$ for all $x \in \mathcal{C}$ (here, $T_{\mathcal{C}}(x)$ is the tangent cone[2] to $\mathcal{C} \subset \mathbb{R}^n$ at $x \in \mathbb{R}^n$).*

The condition that $\tilde{F}(x) \in T_{\mathcal{C}}(x)$ for all $x \in \mathcal{C}$ is called the *sub-tangentiality condition*, and can be enforced using the constraints of an optimization-based feedback controller of the form (1). We show how in the following. Suppose that $\mathcal{C}$ is parameterized as $\mathcal{C} = \{x \in \mathbb{R}^n \mid h_j(x) \geq 0,\ 1 \leq j \leq p\}$, where $h_j : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable for $j = 1, \ldots, p$, and the dynamics take the form

$$\dot{x} = F(x,u) = F_0(x) + \sum_{i=1}^{m} u_i F_i(x), \qquad (27)$$

for smooth functions $F_i : \mathbb{R}^n \to \mathbb{R}^n$ for $i \in \{0, \ldots, m\}$. Next, define $A(x) \in \mathbb{R}^{p\times m}$ and $b(x) \in \mathbb{R}^p$ as

$$A(x) = \begin{bmatrix} \mathcal{L}_{F_1} h_1(x) & \ldots & \mathcal{L}_{F_m} h_1(x) \\ \vdots & \ddots & \vdots \\ \mathcal{L}_{F_1} h_p(x) & \ldots & \mathcal{L}_{F_m} h_p(x) \end{bmatrix},$$

$$b(x) = \begin{bmatrix} -\alpha(h_1(x)) - \mathcal{L}_{F_0} h_1(x) \\ \vdots \\ -\alpha(h_m(x)) - \mathcal{L}_{F_0} h_m(x) \end{bmatrix},$$

where α is a class-$\mathcal{K}$ function. Let $A_j(x)$ denote the jth row of $A(x)$, and for $J \subset \{1, \ldots, p\}$, let $A_J(x)$ denote the matrix consisting of the rows of $A(x)$ corresponding to $j \in J$.

In the literature on optimization-based control design [11], the feasibility of the system $A_j(x)u \geq b_j(x)$ for all $x \in \mathbb{R}^n$ such that $h_j(x) \geq 0$ is equivalent to h_j being a *control barrier function* for the set $\{x \in \mathbb{R}^n \mid h_j(x) \geq 0\}$. Since we are considering the case where $\mathcal{C}$ is possibly parameterized by multiple inequalities, here we make the stronger assumption that the system $A(x)u \geq b(x)$ (where the inequality holds component-wise) is feasible for all $x \in \mathcal{C}$. In this case, if $\mathcal{C}$ satisfies an appropriate constraint qualification condition (e.g., MFCQ or LICQ) and $u^* : \mathbb{R}^n \to \mathbb{R}^m$ is a feedback controller such that $A(x)u^*(x) \geq b(x)$ for all $x \in \mathcal{C}$, then the closed-loop dynamics satisfies the sub-tangentiality condition $F(x, u^*(x)) \in T_{\mathcal{C}}(x)$. Such a controller can be obtained from the solution of

[2]Recall that $T_{\mathcal{C}}(x) = \{v \in \mathbb{R}^n \mid \liminf_{h\to 0} \frac{\mathrm{dist}(x+hv,\mathcal{C})}{h} = 0\}$.

a parametric optimization problem of the form (1) where $g(x, u) = b(x) - A(x)u$.

To show invariance invoking Theorem 5.1, one needs to additionally ensure that the closed-loop dynamics has unique solutions. The conditions discussed in Section 4 and summarized in Table 1 can be translated into easily checkable conditions on the objective function, the matrix $A(x)$, and the vector $b(x)$. The following result uses [54, Theorem 3.6] to ensure uniqueness, and therefore forward invariance.

Theorem 5.2. (Sufficient conditions for forward invariance with respect to closed-loop dynamics): *Consider the dynamics* (27) *and the optimization problem* (1) *where $f \in \mathcal{C}^{1,2}(\mathbb{R}^n \times \mathbb{R}^m)$ is strictly convex, and $g(x, u) = b(x) - A(x)u$. Assume*

- *For all $x \in \mathcal{C}$, there exists $u \in \mathbb{R}^m$ such that $A(x)u > b(x)$, and* (1) *has at least one minimizer.*
- *For all $x \in \mathcal{C}$, there is an open set $U_x \subset \mathbb{R}^n$ containing x such that, for any subset J in $\{1, \ldots, p\}$, the matrix $A_J(y)$ has constant rank for all $y \in U_x$.*

Then the closed-loop system under the optimization-based controller (1) *has unique solutions, and $\mathcal{C}$ is forward invariant.*

In the case where the closed-loop dynamics are point-Lipschitz, solutions are not necessarily unique and therefore forward invariance of $\mathcal{C}$ cannot be guaranteed by Theorem 5.1. In fact, the following is an example of a system where the sub-tangentiality condition holds but there exist solutions starting in $\mathcal{C}$ that eventually leave.

Example 5.3. (Point-Lipschitz differential equation violating forward invariance): Let $\mathcal{C} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \leq 0\}$ and consider the system with the feedback controller defined in Example 4.2. Because $\mathcal{C}$ satisfies LICQ, the tangent cone can be computed as $T_{\mathcal{C}}(x_1, x_2) = \mathbb{R}^2$ if $x_2 < 0$, and $T_{\mathcal{C}}(x_1, 0) = \{(\xi_1, \xi_2) \mid \xi_2 \leq 0\}$. The closed-loop system satisfies $F(x, u^*(x)) = (\frac{1}{2}, u_4^*(x_1, x_2)) \in T_{\mathcal{C}}(x_1, x_2)$ for all $(x_1, x_2) \in \mathcal{C}$. However, the solution $y_2(t) = (\frac{1}{2}t, \frac{1}{8}t^2)$ satisfies $y_2(0) \in \mathcal{C}$ and $y_2(t) \notin \mathcal{C}$ for all $t > 0$. •

Example 4.2 is problematic because it shows that even if the *sub-tangentiality* condition for a safe set $\mathcal{C}$ is included as one of the constraints of the optimization-based controller, if the solutions of the closed-loop system are not unique, some of the solutions might leave the safe set $\mathcal{C}$. However, using the notion of minimal barrier functions [74], the following result gives a condition for forward invariance that can be applied to systems with non-unique solutions.

Theorem 5.4. (Minimal Barrier Functions, [74, Theorem 1]): *Let $\tilde{F} : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and consider the system $\dot{x} = \tilde{F}(x)$. Let $h : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function and let $\mathcal{C} = \{x \in \mathbb{R}^n \mid h(x) \geq 0\}$ be a nonempty set. If h is a minimal barrier function, cf. [74, Definition 2], then any solution of $\dot{x} = \tilde{F}(x)$ with initial condition in $\mathcal{C}$ remains in $\mathcal{C}$ for all positive times.*

A simple scenario in which h is a minimal barrier function is if there exists a strictly increasing function $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$ and an open set $\mathcal{D}$ with $\mathcal{C} \subset \mathcal{D}$ such that $\nabla h(x)^\top \tilde{F}(x) \geq -\alpha(h(x))$ for all $x \in \mathcal{D}$. Such a set $\mathcal{D}$ and class $\mathcal{K}$ function α do not exist in Example 5.3. Since Theorem 5.4 only requires $\tilde{F}$ to be continuous, the system $\dot{x} = \tilde{F}(x)$ might have multiple solutions starting from the same initial condition. However, the result ensures that if the initial condition is in $\mathcal{C}$, then all solutions remain in $\mathcal{C}$ for all positive times. Moreover, since point-Lipschitz functions are continuous, Theorem 5.4 can be applied to differential equations defined by point-Lipschitz functions. Therefore, if one of the constraints in (1) corresponds to the minimal control barrier function condition of a function h, and if the resulting controller is point-Lipschitz (e.g., by satisfying the hypothesis of Proposition 3.2), then all solutions of the closed-loop system that start in $\mathcal{C} := \{x \in \mathbb{R}^n \mid h(x) \geq 0\}$ remain in $\mathcal{C}$ for all positive times.

Finally, we also note that if u^* is discontinuous, the closed-loop system might not have unique solutions and hence the assumptions of Theorem 5.1 will not hold. Therefore, this result cannot be used to guarantee forward invariance of sets. However, the following result gives a sufficient condition for forward invariance of sets under Filippov solutions. It follows as an adaptation of [75, Theorem 1], which gives a sufficient condition for forward invariance of sets under hybrid systems.

Theorem 5.5. (Forward invariance under Filippov solutions of closed-loop dynamics): *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function and $\mathcal{C} := \{x \in \mathbb{R}^n \mid h(x) \geq 0\}$. Further let $\mathcal{D} \subset \mathbb{R}^n$ be a set* *containing $\mathcal{C}$ such that, for each initial condition x_0 in $\mathcal{D}$, there exists a forward complete Filippov solution of* (18) *with initial condition at x_0. Let $\mathcal{P}(\mathbb{R}^n)$ denote the collection of subsets of $\mathbb{R}^n$ and let $\mathcal{F} : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be the Filippov set-valued map of* (18)*, i.e.,*

$$\mathcal{F}(x) := \bigcap_{\delta>0} \bigcap_{\mu(S)=0} \overline{co} \left\{ \bigcup_{y \mid \|y-x\| \le \delta} F(y, u^*(y)) \backslash S \right\},$$

where $\overline{co}$ denotes the convex closure and μ denotes the Lebesgue measure. Further assume that there exists a neighborhood $\mathcal{U}_f$ of $\partial\mathcal{C} = \{x \in \mathbb{R}^n \mid h(x) = 0\}$ such that

$$\nabla h(x)^T \eta \ge 0, \ \forall x \in \mathcal{U}_f \backslash \mathcal{C} \text{ and } \forall \eta \in \mathcal{F}(x). \quad (28)$$

Then, all Filippov solutions of (18) *with initial condition at $\mathcal{C}$ remain in $\mathcal{C}$ for all positive times.*

In particular, Theorem 5.5 ensures that under the assumptions of Proposition 4.5, and if Filippov solutions are defined for all positive times, then the *sub-tangentiality*-like condition (28) guarantees forward invariance of Filippov solutions. We note also that Theorem 5.5 is possibly conservative, and tighter conditions that guarantee forward invariance for Filippov solutions could be developed using an adapted notion of minimal barrier functions for discontinuous systems.

6. Conclusions and Outlook

We have provided an integrative presentation of insights and results about the regularity properties of optimization-based controllers, and their implication in different properties of interest of control systems.

Regularity properties: Under appropriate constraint qualifications and conditions on the data that defines the optimization problem, we have shown that optimization-based controllers are locally Lipschitz, continuously differentiable, and even analytic. We have also characterized the properties enjoyed by parametric optimizers arising from optimization problems defined by second-order continuously differentiable objective function and constraints, strictly (or strongly) convex objective function, and feasible set with nonempty interior (the same properties as in Robinson's counterexample). We have shown that, even though such parametric optimizers might not be locally Lipschitz, they enjoy other important regularity properties, like point-Lipschitz continuity. Even if the optimization-based controller is discontinuous, under appropriate conditions on the optimization problem data, we have shown that it is measurable and locally bounded.

Implications on the resulting closed-loop systems: When our results are applied to the motivating examples in Section 1, they improve upon the existing results in the literature by providing a more detailed description of the regularity of the corresponding controller under a wider range of conditions. Building on the results on regularity properties of optimization-based controllers, we have studied the existence and uniqueness of classical and Filippov solutions of closed-loop systems obtained from an optimization-based controller, and identified conditions ensuring that all (not necessarily unique) solutions remain in a safe set of interest.

Outlook: The results presented in this work show that the regularity properties of optimization-based controllers are determined by the smoothness/convexity and constraint qualification properties of the optimization problems defining them. This opens the door to the possibility of designing optimization problems with the appropriate conditions and constraint qualification properties in order to endow the associated optimization-based controller with certain desired regularity properties. For example, in the context of safety-critical control, it is sufficient to find safe sets and control barrier functions for those sets to guarantee that an associated control barrier function based controller is locally Lipschitz. In certain cases, one can guarantee that these control objectives are obtained without continuity or even uniqueness of solutions to the resulting closed-loop systems. Characterizing conditions on the objective function and constraints to ensure that control objectives are achieved even in the absence of local Lipschitz continuity is an important direction for future work. In particular, understanding one-sided Lipschitzness in the context of parametric optimization is important in the context of safety-critical control, since uniqueness of solutions allows one to verify forward-invariance of a safe set via Nagumo's Theorem. Besides the examples mentioned in Section 1, the relevance of the results presented herein also applies to other areas of systems and control, where controllers need to be designed with desirable regularity properties, such as backstepping [76], where virtual controllers need to be differentiated at the intermediate layers to construct a composite Lyapunov or barrier [49] function for the composite system. Other ideas for future work include improving further the results presented in this paper to more specific classes of problems, such as CBF-based quadratic programs or second-order convex programs, as well as MPC-based controllers. We also plan to relax the differentiability and convexity assumptions considered throughout this paper and give regularity results for set-valued optimizers using the theory of continuous selections [77].

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