Functional Observers for Output-Feedback Safety with High-Order Control Barrier Functions

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***Abstract*—This letter studies output-feedback safety filtering for linear systems using high-order control barrier functions (HOCBFs) when the full state is unavailable. Since the HOCBF conditions depend only on specific state functionals, full-state reconstruction is unnecessary; instead, their implementation requires only functional observability or detectability of these quantities. We show that the HOCBF gains determine the required functionals and hence their observability properties, coupling safety and estimation design. We exploit this structure using scalar functional observers and develop a polynomial co-design framework that characterizes HOCBF observability and detectability, observer existence and order, and convergence-rate limitations. A certified one-sided estimation-error bound then enables safe output-feedback implementation.**

***Index Terms*—High-order control barrier functions, functional observers, output-feedback control, safety filters.**

I. Introduction

Safety-critical control enforces state constraints while preserving a nominal control objective. Control Barrier Functions (CBFs) express safety through forward invariance and incorporate constraints into optimization-based controllers [1], [2]. Exponential and High-Order CBFs (HOCBFs) extend this framework to higher-relative-degree constraints [3], [4]. Their implementation typically requires state-dependent quantities, often motivating a full-state observer when only outputs are measured. Full-state reconstruction, however, may be unnecessary or even impossible for unobservable systems. This raises the central question: what state information is actually required by an HOCBF safety filter, and how should the HOCBF gains (i.e., the class $\mathcal{K}$ functions associated with the HOCBF construction) be chosen to make that information estimable?

Several approaches extend CBF-based safety to partial or uncertain state information. Measurement-robust CBFs account for state-estimation or perception errors, including high-relative-degree constraints [5]–[7], while observer-based methods incorporate estimation-error bounds into barrier constraints [8], [9]. Related work considers robust output CBFs, joint observer–barrier synthesis, and backup CBFs under state uncertainty and bounded inputs [10]–[12]. Most closely related, [13] studies output-feedback control of LTI systems with operational constraints using a full-state Luenberger observer and provides a simple condition ensuring that safety is maintained after a transient despite state-estimation error. These methods typically reconstruct the state and then account for estimation error, rather than characterize the weaker requirement of estimating only the functionals needed by the HOCBF inequalities. Thus, they do not identify when output-feedback safety remains possible without full-state reconstruction.

Functional observers estimate prescribed state functionals and may exist even when the full system is unobservable. Their existence, functional observability and detectability, reduced-order construction, and assignable error dynamics are well studied [14]–[20]. In this literature, however, the functional is fixed. In an HOCBF safety filter, it depends on the HOCBF gains, which are design parameters. Gain selection therefore affects both the barrier condition and whether the required functional is observable or detectable, raising the question of how to select the gains to ensure estimability with fast convergence while maintaining a large admissible set.

We study this problem for linear systems with affine state constraints. Recent work has characterized feasibility, compatibility, explicit implementation, and closed-loop dynamics of safety filters in this setting [21]–[23], but output-feedback design remains largely unexplored. HOCBF theory for general control-affine systems has also studied feasibility and conservatism [4], [24], but not output feedback from a functional-observability perspective or how the HOCBF gains affect observer existence, order, and convergence. This coupling between HOCBF design and functional-observer dynamics therefore remains uncharacterized.

This work bridges this gap for LTI systems with affine safety constraints. First, we construct an output-feedback safety filter that estimates only the scalar functionals required by the HOCBF constraints and uses a certified one-sided error bound to guarantee safety. Second, we co-design the HOCBF gains and functional observers, characterizing HOCBF observability and detectability for fixed gains, when these properties are achievable by gain selection, and the resulting observer-order and convergence-rate limits, even when the full state is unobservable. Third, after selecting the gain values and observer, we exploit the remaining gain-ordering freedom to enlarge the HOCBF admissible set without changing the final HOCBF inequality or observer design.

II. Problem Formulation

This section formulates the output-feedback safety problem and introduces the functional-observer framework.

Notation: Let $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ denote the real and nonnegative real numbers, and I_k the $k \times k$ identity. For a matrix M, $\ker(M)$, $\mathrm{row}(M)$, and $M_{i,:}$ denote its kernel, row space, and ith row. For a subspace $\mathcal{V}$, $\mathcal{V}^{\perp}$ denotes its orthogonal complement. For polynomials p, q, $p \mid q$, $\gcd(p, q)$, and $\deg(p)$ denote divisibility, the monic greatest common divisor, and polynomial degree, respectively, with multiplicities included. The Euclidean norm is $\|\cdot\|$, and vector inequalities are componentwise.

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A. High-Order Affine Control Barrier Functions

Consider the linear time-invariant system

$$\dot{x} = Ax + Bu, \qquad y = Cx, \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^q$. Only the output y is available to the controller.

Safety is specified by the affine constraints

$$h_i(x) := f_i^\top x + d_i \geq 0, \qquad i = 1, \dots, p,$$

with safe set $\mathcal{S} := \bigcap_{i=1}^{p} \{x \in \mathbb{R}^n : h_i(x) \geq 0\}$. Assume that h_i has relative degree r_i, that is,

$$f_i^\top A^k B = 0, \quad k = 0, \dots, r_i - 2, \qquad f_i^\top A^{r_i - 1} B \neq 0,$$

where the first condition is omitted when $r_i = 1$.

For each constraint, define the ordered gain vector

$$\boldsymbol{\alpha}_i := (\alpha_{i1}, \dots, \alpha_{ir_i}), \qquad \alpha_{ij} > 0,$$

and the associated HOCBF polynomial

$$\phi_i(s) := \prod_{j=1}^{r_i} (s + \alpha_{ij}).$$

The polynomial ϕ_i depends only on the selected gain values, whereas $\boldsymbol{\alpha}_i$ also specifies their order in the HOCBF recursion

$$\psi_{i,0} := h_i, \qquad \psi_{i,j} := \dot{\psi}_{i,j-1} + \alpha_{ij}\psi_{i,j-1}, \quad j = 1, \dots, r_i, \tag{2}$$

where the derivatives are evaluated along (1). The functions up to order $r_i - 1$ define the HOCBF admissible sets

$$\mathcal{C}_{i,\boldsymbol{\alpha}_i} := \bigcap_{j=0}^{r_i - 1} \{x : \psi_{i,j}(x) \geq 0\}, \qquad \mathcal{C}_{\boldsymbol{\alpha}} := \bigcap_{i=1}^{p} \mathcal{C}_{i,\boldsymbol{\alpha}_i}, \tag{3}$$

where $\boldsymbol{\alpha} := (\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_p)$. Since $\psi_{i,0} = h_i$, one has $\mathcal{C}_{\boldsymbol{\alpha}} \subseteq \mathcal{S}$.

For system (1), the final HOCBF conditions $\psi_{i,r_i} \geq 0$ yield

$$L_\phi x + Du + b_\phi \geq 0, \tag{4}$$

where $L_\phi \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$, and $b_\phi \in \mathbb{R}^p$ are defined rowwise by

$$[L_\phi]_{i,:} := f_i^\top \phi_i(A), \quad [D]_{i,:} := f_i^\top A^{r_i - 1} B, \quad [b_\phi]_i := d_i \phi_i(0).$$

By the standard HOCBF result [4], any locally Lipschitz input satisfying (4) renders $\mathcal{C}_{\boldsymbol{\alpha}}$ forward invariant for $x(0) \in \mathcal{C}_{\boldsymbol{\alpha}}$.

For each $i = 1, \dots, p$, define the scalar HOCBF functional

$$\ell_i := f_i^\top \phi_i(A), \qquad z_i := \ell_i x,$$

and collect these quantities as

$$z := (z_1, \dots, z_p)^\top = L_\phi x.$$

Hence, (4) becomes $z + Du + b_\phi \geq 0$, so the HOCBF constraints require only z, rather than the full state x.

For a nominal output-feedback controller $u_{\mathrm{nom}}(y, t)$, consider the safety filter

$$\begin{aligned} u_{\mathrm{sf}}(y, z, t) := \arg\min_{u \in \mathbb{R}^m} \quad & \frac{1}{2}\|u - u_{\mathrm{nom}}(y, t)\|^2 \\ \text{s.t.} \quad & z + Du + b_\phi \geq 0. \end{aligned} \tag{5}$$

When feasible, the strictly convex objective gives a unique minimizer. Feasibility is assumed throughout; full row rank of D is sufficient for arbitrary right-hand sides.

The gain values $\boldsymbol{\alpha}$ and their ordering play different roles. The polynomial ϕ_i determines ℓ_i and $[b_\phi]_i$, whereas D is gain-independent. Permuting the same gain values $\boldsymbol{\alpha}_i$ therefore leaves the final HOCBF inequality and (5) unchanged, but may change $\mathcal{C}_{i,\boldsymbol{\alpha}_i}$.

B. Functional-Observer Preliminaries

This subsection summarizes the functional-observer preliminaries needed for the co-design analysis.

For $M \in \mathbb{R}^{s \times n}$ and $k \geq 0$, define

$$\mathcal{O}_k(M, A) := \begin{bmatrix} M^\top & (MA)^\top & \cdots & (MA^k)^\top \end{bmatrix}^\top.$$

The corresponding unobservable subspace is $\mathcal{U}(M) := \ker \mathcal{O}_{n-1}(M, A)$. In particular, $\mathcal{U}(C)$ is the output-unobservable subspace: two initial states driven by the same input generate the same output trajectory if and only if their initial-state difference lies in $\mathcal{U}(C)$.

Definition 1. *[Functional observability and detectability] A scalar functional $z = \bar{\ell}x$ is* functionally observable *from $y = Cx$ if any two initial states that produce the same output trajectory under the same input also produce the same z trajectory. It is* functionally detectable *if the difference between their z trajectories converges to zero [19], [20].*

For system (1), equivalently,

$$\begin{aligned} &z \text{ functionally observable} \iff \mathcal{U}(C) \subseteq \mathcal{U}(\bar{\ell}), \\ &z \text{ functionally detectable} \iff \lim_{t \to \infty} \bar{\ell} e^{At} v = 0, \quad \forall v \in \mathcal{U}(C). \end{aligned}$$

Functional observability implies functional detectability, but not conversely. Both properties may hold even when (A, C) is not observable.

We now specialize these notions to HOCBFs, distinguishing a fixed gain choice from the existence of a suitable gain choice.

Definition 2. *[HOCBF observability and detectability for fixed gains] For a fixed gain vector $\boldsymbol{\alpha}_i = (\alpha_{i1}, \dots, \alpha_{ir_i})$ with $\alpha_{ij} > 0$, the pair $(h_i, \boldsymbol{\alpha}_i)$ is* HOCBF-observable *(resp.,* HOCBF-detectable*) from y if the corresponding HOCBF functional $z_i = \ell_i x = f_i^\top \phi_i(A) x$ is functionally observable (resp., functionally detectable) from y.*

Definition 3. *[HOCBF-observable and detectable constraints] The constraint h_i is* HOCBF-observable *(resp.,* HOCBF-detectable*) from y if there exists a positive gain vector $\boldsymbol{\alpha}_i$ such that $(h_i, \boldsymbol{\alpha}_i)$ is HOCBF-observable (resp., HOCBF-detectable) from y.*

We also define

$$\kappa_C := \min\left\{k \geq 0 : \operatorname{row} \mathcal{O}_k(C, A) = \mathcal{U}(C)^\perp\right\}. \tag{6}$$

Hence, κ_C is the smallest k for which $\mathcal{O}_k(C, A)$ captures all state information available from the output trajectory.

To estimate z_i directly, consider the scalar functional observer [15], [19], [20]

$$\begin{aligned} \dot{\eta}_i &= N_i \eta_i + J_i y + G_i u, \\ \hat{z}_i &= R_i \eta_i + E_i y, \end{aligned} \tag{7}$$

where $\eta_i \in \mathbb{R}^{\nu_i}$ and ν_i is the observer order. The observer matrices can be obtained using standard functional-observer synthesis methods [15], [18].

We consider functional observers whose estimation errors admit autonomous linear dynamics. Specifically, suppose there exists $T_i \in \mathbb{R}^{\nu_i \times n}$ satisfying

$$T_iA = N_iT_i + J_iC, \quad G_i = T_iB, \quad \ell_i = R_iT_i + E_iC, \quad (8)$$

as in standard functional-observer constructions [17].

Define the observer error state and estimation error as

$$\varepsilon_i := T_ix - \eta_i, \qquad e_i := z_i - \hat{z}_i.$$

Using the system and observer dynamics, $\dot{\varepsilon}_i = N_i\varepsilon_i + (T_iA - N_iT_i - J_iC)x + (T_iB - G_i)u$. Hence, (8) gives

$$\dot{\varepsilon}_i = N_i\varepsilon_i, \qquad e_i = R_i\varepsilon_i. \quad (9)$$

Because u is available to the observer, its contribution can be exactly reproduced by choosing $G_i = T_iB$; thus, this condition places no additional restriction on B.

We take the error realization to be minimal and define

$$\beta_i(s) := \det(sI_{\nu_i} - N_i),$$

with $\beta_i = 1$ when $\nu_i = 0$.

The following existence condition is standard [18].

Proposition 1. *[Functional-observer existence] Fix ν_i and a real monic polynomial β_i of degree ν_i. An observer satisfying* (8) *with error polynomial β_i exists if and only if*

$$\ell_i\beta_i(A) \in \operatorname{row} \mathcal{O}_{\nu_i}(C, A).$$

For the minimal realization (9)*, $e_i(t) \to 0$ for all initial errors if and only if β_i is Hurwitz.*

C. Output-Feedback Safety Design Problem

The filter (5) requires the HOCBF functionals z_i. Since $\ell_i = f_i^\top \phi_i(A)$, these functionals depend on the HOCBF gains, coupling HOCBF and observer design.

Problem 1. *[] For system* (1)*, implement* (5) *from output measurements using the functional observers* (7) *while guaranteeing safety. Characterize HOCBF observability and detectability, observer existence, order, and rate under gain selection, and determine the ordering maximizing $\mathcal{C}_{\boldsymbol{\alpha}}$.*

III. Safe Functional-Observer Filtering

This section derives a certified error bound for output-feedback HOCBF safety filtering.

Let

$$\hat{z} := (\hat{z}_1, \dots, \hat{z}_p)^\top, \qquad e := z - \hat{z}.$$

If $e_i < 0$, then $\hat{z}_i$ overestimates the true HOCBF functional z_i and may compromise safety, whereas $e_i > 0$ is conservative. Hence, only a lower error bound is needed. Suppose a known function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}^p_{\geq 0}$ satisfies

$$e(t) \geq -\rho(t), \qquad t \geq 0. \quad (10)$$

Using this bound, replace the safety filter (5) by the implementable output-feedback filter

$$\begin{aligned} \widetilde{u}(y, \hat{z}, t) := \arg\min_{u \in \mathbb{R}^m} \quad & \frac{1}{2}\|u - u_{\mathrm{nom}}(y, t)\|^2 \\ \text{s.t.} \quad & \hat{z} + Du + b_\phi \geq \rho(t). \end{aligned} \quad (11)$$

Under the feasibility assumption of Section II-A, the strictly convex objective gives a unique minimizer. The next result shows that the tightening by ρ preserves safety.

Proposition 2. *[Safety under a one-sided error bound] Suppose that* (10) *holds and the solution of* (11) *is locally Lipschitz. Then $\mathcal{C}_{\boldsymbol{\alpha}}$ is forward invariant.*

Proof. Since $e = z - \hat{z}$, the constraint in (11) and (10) give

$$z + Du + b_\phi = \hat{z} + Du + b_\phi + e \geq \rho - \rho = 0.$$

Thus, $\widetilde{u}$ satisfies the true HOCBF inequality (4). Forward invariance follows from the standard HOCBF result [4]. □

The margin $\rho_i(t)$ is minimal given (10): any smaller margin would allow $e_i(t) = -\rho_i(t)$ to violate the true HOCBF inequality.

We next construct such a bound for the autonomous error dynamics in Section II-B. Define $\chi := \begin{bmatrix} \varepsilon_1^\top & \cdots & \varepsilon_p^\top \end{bmatrix}^\top$, $A_e := \operatorname{blkdiag}(N_1, \dots, N_p)$, and $C_e := \operatorname{blkdiag}(R_1, \dots, R_p)$. Then

$$\dot{\chi} = A_e\chi, \qquad e = C_e\chi. \quad (12)$$

Assume

$$\chi(0)^\top P\chi(0) \leq \bar{V}_0, \quad (13)$$

where $P \succ 0$ and $\bar{V}_0 \geq 0$ are known.

For each $i = 1, \dots, p$, define

$$q_i(t) := [C_e]_{i,:}e^{A_et}, \quad \rho_i(t) := \sqrt{\bar{V}_0\, q_i(t)P^{-1}q_i(t)^\top}, \quad (14)$$

and let $\rho := (\rho_1, \dots, \rho_p)^\top$. The following result certifies the required error bound.

Proposition 3. *[Componentwise estimation-error bound] Under* (12) *and* (13)*, ρ satisfies* (10)*. If A_e is Hurwitz, then $\rho(t) \to 0$ exponentially. Moreover, if D has full row rank,*

$$\widetilde{u}(y, \hat{z}, t) - u_{\mathrm{sf}}(y, z, t) \to 0.$$

Proof. From (12), $e_i(t) = q_i(t)\chi(0)$. By weighted Cauchy–Schwarz and (13), $|e_i(t)| \leq \sqrt{\bar{V}_0\, q_i(t)P^{-1}q_i(t)^\top} = \rho_i(t)$. Hence, $e_i(t) \geq -\rho_i(t)$ for all i, proving (10). For fixed i, t with $q_i(t) \neq 0$, the lower bound is attained by an initial error on the boundary of (13) in the direction $-P^{-1}q_i(t)^\top$, so it is pointwise tight.

If A_e is Hurwitz, then $e(t)$ and $\rho(t)$ converge to zero exponentially. Thus, $\hat{z}(t) \to z(t)$ and the constraint in (11) converges to that in (5). Under the full-row-rank assumption on D, continuity of the unique QP solution then gives $\widetilde{u}(y, \hat{z}, t) - u_{\mathrm{sf}}(y, z, t) \to 0$. □

IV. HOCBF Gain–Observer Co-Design

This section co-designs HOCBF gains and functional observers through unobservable-factor allocation, yielding observability, detectability, order, and rate results.

A. Unobservable-Factor Allocation

This subsection identifies the output-unobservable dynamics relevant to each safety constraint and determines which factors can be removed by the HOCBF polynomial and which must appear in the observer-error polynomial.

Consider two trajectories x_1 and x_2 of (1) driven by the same input and producing the same output trajectory. Let $\delta x := x_1 - x_2$. Then $\delta x(t) \in \mathcal{U}(C)$. Let $n_u := \dim \mathcal{U}(C)$ and let $V_{\mathcal{U}} \in \mathbb{R}^{n\times n_u}$ have columns forming a basis for $\mathcal{U}(C)$. Since $\mathcal{U}(C)$ is A-invariant, there exists $A_{\mathcal{U}} \in \mathbb{R}^{n_u\times n_u}$ with

$$AV_{\mathcal{U}} = V_{\mathcal{U}} A_{\mathcal{U}}. \tag{15}$$

Writing $\delta x = V_{\mathcal{U}}\xi$ gives $\dot{\xi} = A_{\mathcal{U}}\xi$.

For constraint i, define

$$\delta h_i := h_i(x_1) - h_i(x_2) = f_i^\top V_{\mathcal{U}}\xi, \qquad c_i := f_i^\top V_{\mathcal{U}}.$$

Thus, c_i determines which output-unobservable dynamics affect the ith constraint.

We define the *relevant unobservable polynomial* μ_i as follows. If $c_i = 0$, set $\mu_i = 1$. Otherwise, let σ_i be the smallest positive integer such that

$$c_i A_{\mathcal{U}}^{\sigma_i} \in \operatorname{span}\{c_i, c_i A_{\mathcal{U}}, \dots, c_i A_{\mathcal{U}}^{\sigma_i-1}\}.$$

Equivalently, for some $a_{i,0}, \dots, a_{i,\sigma_i-1}$,

$$c_i A_{\mathcal{U}}^{\sigma_i} + \sum_{k=0}^{\sigma_i-1} a_{i,k} c_i A_{\mathcal{U}}^{k} = 0. \tag{16}$$

Define

$$\mu_i(s) := s^{\sigma_i} + \sum_{k=0}^{\sigma_i-1} a_{i,k} s^k. \tag{17}$$

Thus, μ_i is obtained from the first linear dependence in the Krylov sequence $c_i, c_i A_{\mathcal{U}}, c_i A_{\mathcal{U}}^2, \dots$ [25]. Equivalently, it is the monic polynomial of smallest degree satisfying

$$f_i^\top V_{\mathcal{U}} \mu_i(A_{\mathcal{U}}) = 0.$$

Hence, μ_i captures exactly the output-unobservable dynamics relevant to the ith constraint. If $A_{\mathcal{U}}$ is diagonalizable, μ_i is the product of the factors $s - \lambda$ corresponding to the distinct unobservable modes affecting the constraint. For nondiagonalizable dynamics, multiplicities account for generalized-mode chains. The polynomial μ_i is independent of the basis $V_{\mathcal{U}}$.

Recall that $\ell_i = f_i^\top \phi_i(A)$. For the two trajectories,

$$\delta z_i := z_i(x_1) - z_i(x_2) = f_i^\top V_{\mathcal{U}} \phi_i(A_{\mathcal{U}})\xi.$$

Thus, common factors of μ_i and ϕ_i, including multiplicities, do not contribute to the output-unobservable part of z_i. Define the remaining unobservable polynomial

$$\mu_i^{\mathrm{rem}} := \frac{\mu_i}{\gcd(\mu_i, \phi_i)}.$$

The following lemma characterizes μ_i and μ_i^{rem}.

Lemma 1. *[Unobservable polynomial characterization] For any polynomials p and q,*

$$f_i^\top V_{\mathcal{U}} p(A_{\mathcal{U}}) = 0 \quad \Longleftrightarrow \quad \mu_i \mid p,$$

and

$$f_i^\top V_{\mathcal{U}} \phi_i(A_{\mathcal{U}}) q(A_{\mathcal{U}}) = 0 \quad \Longleftrightarrow \quad \mu_i^{\mathrm{rem}} \mid q. \tag{18}$$

Proof. For the first equivalence, $\mu_i \mid p$ implies the result by the definition of μ_i. Conversely, write $p = q_1\mu_i + r$ with $\deg(r) < \deg(\mu_i)$. Then $f_i^\top V_{\mathcal{U}} r(A_{\mathcal{U}}) = 0$, so minimality of μ_i gives $r = 0$.

For the second, apply the first equivalence to $p = \phi_i q$:

$$\mu_i \mid \phi_i q \quad \Longleftrightarrow \quad \mu_i^{\mathrm{rem}} \mid q.$$

□

Thus, μ_i^{rem} captures the unobservable dynamics remaining in z_i.

We can now state the main polynomial co-design result.

Theorem 1. *[Unobservable-factor allocation] Fix $\boldsymbol{\alpha}_i$. Every scalar functional observer satisfying* (8) *must satisfy*

$$\mu_i^{\mathrm{rem}} \mid \beta_i. \tag{19}$$

Conversely, if $\nu_i \geq \kappa_C$, every real monic polynomial β_i of degree ν_i satisfying (19) *is realizable by an order-ν_i functional observer satisfying* (8)*.*

Proof. By Proposition 1, $\ell_i\beta_i(A) \in \operatorname{row}\mathcal{O}_{\nu_i}(C, A) \subseteq \mathcal{U}(C)^\perp$. Hence, $\ell_i\beta_i(A)V_{\mathcal{U}} = 0$. Using $\ell_i = f_i^\top\phi_i(A)$ and (15) gives $f_i^\top V_{\mathcal{U}}\phi_i(A_{\mathcal{U}})\beta_i(A_{\mathcal{U}}) = 0$. Equation (18) then gives $\mu_i^{\mathrm{rem}} \mid \beta_i$.

Conversely, if $\mu_i^{\mathrm{rem}} \mid \beta_i$, then (18) gives $\ell_i\beta_i(A)V_{\mathcal{U}} = 0$, and hence $\ell_i\beta_i(A) \in \mathcal{U}(C)^\perp$. For $\nu_i \geq \kappa_C$, $\mathcal{U}(C)^\perp = \operatorname{row}\mathcal{O}_{\nu_i}(C, A)$, so Proposition 1 gives the desired observer. □

The residual polynomial characterizes both fixed-gain and constraint-level HOCBF observability and detectability.

Corollary 1. *[HOCBF observability and detectability]*

1) *For fixed $\boldsymbol{\alpha}_i$, $(h_i, \boldsymbol{\alpha}_i)$ is HOCBF-observable if and only if $\mu_i^{\mathrm{rem}} = 1$ (equivalently, $\mu_i \mid \phi_i$), and HOCBF-detectable if and only if μ_i^{rem} is Hurwitz (equivalently, μ_i is Hurwitz).*
2) *h_i is HOCBF-observable if and only if all roots of μ_i are real and strictly negative with $\deg(\mu_i) \leq r_i$, and HOCBF-detectable if and only if μ_i is Hurwitz.*

Proof. For fixed $\boldsymbol{\alpha}_i$, Definition 2 and (18) give observability if and only if $\mu_i^{\mathrm{rem}} = 1$. Detectability holds if and only if the remaining unobservable modes decay, i.e., μ_i^{rem} is Hurwitz. Since ϕ_i has only real strictly negative roots, removing common factors with ϕ_i cannot remove a non-Hurwitz root; hence μ_i^{rem} is Hurwitz if and only if μ_i is Hurwitz.

By Definition 3, h_i is HOCBF-observable if and only if some admissible ϕ_i contains μ_i. Since ϕ_i has r_i real strictly negative roots, this holds exactly when all roots of μ_i are real and strictly negative and $\deg(\mu_i) \leq r_i$. Detectability follows from its gain independence. □

Thus, HOCBF observability requires μ_i to have only real negative roots and $\deg(\mu_i) \leq r_i$, whereas HOCBF detectability is gain-independent and requires only that μ_i be Hurwitz.

Theorem 1 also gives $\nu_i \geq \deg(\mu_i^{\mathrm{rem}})$, while $\nu_i = \max\{\deg(\mu_i^{\mathrm{rem}}), \kappa_C\}$ is always sufficient. This is therefore the

Algorithm 1 HOCBF–observer co-design

1: Compute $c_i = f_i^\top V_{\mathcal{U}}$ and obtain μ_i from (16).
2: Select up to r_i real negative roots ζ of μ_i, counting multiplicity, and set $\alpha_{ij} = -\zeta$. To achieve HOCBF observability, assign all factors when permitted by Corollary 1. Choose any remaining gains positive.
3: Compute $\mu_i^{\mathrm{rem}} = \mu_i / \gcd(\mu_i, \phi_i)$.
4: Choose $\nu_i = \max\{\deg(\mu_i^{\mathrm{rem}}), \kappa_C\}$ and a real monic β_i such that $\mu_i^{\mathrm{rem}} \mid \beta_i$. If vanishing estimation error is required, μ_i must be Hurwitz; choose any remaining roots of β_i as desired, e.g., to meet a specified convergence rate.
5: Construct the observer matrices satisfying (8) using a standard functional-observer synthesis method [15], [18].

natural default choice: larger orders provide additional freely assignable factors in β_i at the cost of a higher-dimensional observer, but do not remove the mandatory factor μ_i^{rem}.

Algorithm 1 summarizes the resulting co-design procedure.

B. Estimation-Rate Allocation

Fix $\boldsymbol{\alpha}_i$ and suppose $(h_i, \boldsymbol{\alpha}_i)$ is HOCBF-detectable. By Corollary 1, μ_i^{rem} is Hurwitz. Theorem 1 requires every root of μ_i^{rem} to be an observer-error pole, thereby limiting the achievable estimation rate. Define

$$\Gamma_i := \begin{cases} -\max\limits_{\mu_i^{\mathrm{rem}}(\zeta)=0} \mathrm{Re}(\zeta), & \mu_i^{\mathrm{rem}} \neq 1, \\ +\infty, & \mu_i^{\mathrm{rem}} = 1. \end{cases}$$

Thus, $\Gamma_i > 0$ when $\mu_i^{\mathrm{rem}} \neq 1$, while $\mu_i^{\mathrm{rem}} = 1$ imposes no mandatory observer-error pole.

An observer achieves exponential rate λ if $|e_i(t)| \leq M e^{-\lambda t}$ for some $M > 0$ over the considered initial errors.

Theorem 2. *[Estimation-rate limit] Fix $\boldsymbol{\alpha}_i$ and suppose $(h_i, \boldsymbol{\alpha}_i)$ is HOCBF-detectable. Every rate $0 \leq \lambda < \Gamma_i$ is achievable by a scalar functional observer of order*

$$\nu_i = \max\{\deg(\mu_i^{\mathrm{rem}}), \kappa_C\}.$$

No scalar functional observer satisfying (8) *can guarantee $\lambda > \Gamma_i$.*

Proof. Let $\lambda < \Gamma_i$. Every root of μ_i^{rem} satisfies $\mathrm{Re}(\zeta) < -\lambda$. Complete μ_i^{rem} to a real monic polynomial β_i of degree $\max\{\deg(\mu_i^{\mathrm{rem}}), \kappa_C\}$ by adding roots with real part more negative than $-\lambda$. Theorem 1 then gives an observer whose error poles satisfy $\mathrm{Re}(\zeta) < -\lambda$.

Conversely, if $\mu_i^{\mathrm{rem}} \neq 1$, Theorem 1 requires $\mu_i^{\mathrm{rem}} \mid \beta_i$. Thus, every admissible error polynomial contains a root with real part $-\Gamma_i$. By minimality of the error realization, this mode affects e_i for some initial error, precluding any guaranteed rate greater than Γ_i. For $\mu_i^{\mathrm{rem}} = 1$, $\Gamma_i = +\infty$ and the upper bound is vacuous. □

C. Set-Maximizing Ordering for Fixed Gains

After the gain values are selected, their ordering remains free. Permuting them preserves ϕ_i, the HOCBF functional, observability properties, and observer design, but may change the HOCBF admissible set.

Let $\boldsymbol{\alpha}_i^{\downarrow}$ and $\boldsymbol{\alpha}_i^{\uparrow}$ denote the gains in nonincreasing and nondecreasing order, respectively, and $\boldsymbol{\alpha}_i^{\pi}$ any other ordering. Denote the corresponding collections over all constraints by $\boldsymbol{\alpha}^{\downarrow}$, $\boldsymbol{\alpha}^{\uparrow}$, and $\boldsymbol{\alpha}^{\pi}$.

Theorem 3. *[Set-maximizing order for fixed gains] For any ordering $\boldsymbol{\alpha}^{\pi}$ of the selected gains,*

$$\mathcal{C}_{\boldsymbol{\alpha}^{\uparrow}} \subseteq \mathcal{C}_{\boldsymbol{\alpha}^{\pi}} \subseteq \mathcal{C}_{\boldsymbol{\alpha}^{\downarrow}}. \tag{20}$$

Proof. Fix constraint i and two orderings differing only by adjacent gains $b \leq a$ at positions k and $k+1$. Let ψ_j and $\widetilde{\psi}_j$ correspond to (b, a) and (a, b), respectively. They coincide for $j < k$, while $\widetilde{\psi}_k - \psi_k = (a-b)\psi_{k-1}$. Thus, $\psi_{k-1}(x), \psi_k(x) \geq 0$ imply $\widetilde{\psi}_k(x) \geq 0$. At level $k+1$, the two recursions coincide because $\left(\frac{d}{dt} + a\right)\left(\frac{d}{dt} + b\right) = \left(\frac{d}{dt} + b\right)\left(\frac{d}{dt} + a\right)$. Since all subsequent gains are identical, all later HOCBF functions also coincide. Thus, moving a larger gain earlier cannot decrease the admissible set. Applying this argument to each constraint, if $x \in \mathcal{C}_{\boldsymbol{\alpha}^{\uparrow}}$, repeatedly moving larger gains earlier yields any $\boldsymbol{\alpha}^{\pi}$, so $x \in \mathcal{C}_{\boldsymbol{\alpha}^{\pi}}$. Likewise, if $x \in \mathcal{C}_{\boldsymbol{\alpha}^{\pi}}$, the same swaps yield $\boldsymbol{\alpha}^{\downarrow}$, so $x \in \mathcal{C}_{\boldsymbol{\alpha}^{\downarrow}}$. Hence, (20) holds. □

For an HOCBF-observable constraint with $\deg(\mu_i) = r_i$, observability fixes $\phi_i = \mu_i$, leaving only the ordering to optimize. Otherwise, free gain values remain; optimizing them changes ϕ_i and the observer design and may require additional gain bounds to admit a finite maximizer.

V. Aircraft Roll–Yaw Example

This section illustrates HOCBF observability, factor allocation, observer order and rate, and gain ordering on a linearized aircraft roll–yaw model with an unobservable full state.

Consider the linearized roll–yaw dynamics of a mid-size aircraft at the operating condition in [26, Sec. 14.8], with velocity 717.17 ft/s, altitude 25,000 ft, and angle of attack $4.5627°$. Using the aileron channel,

$$\dot{x} = Ax + Bu, \qquad x = \begin{bmatrix} \beta_{\mathrm{s}} & p_{\mathrm{s}} & r_{\mathrm{s}} \end{bmatrix}^\top,$$

$$A = \begin{bmatrix} -0.11794 & 0.000848 & -1.0001 \\ -7.0113 & -1.4492 & 0.22059 \\ 6.3035 & 0.065114 & -0.41172 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -7.9662 \\ 0.60926 \end{bmatrix}.$$

Here, β_{s} is the sideslip angle and p_{s} and r_{s} are the stability-axis roll and yaw rates.

The matrix A has a stable real eigenvalue $\lambda_{\mathrm{u}} \approx -1.3902$. Let v_{u} be a corresponding eigenvector and choose C such that $\ker(C) = \mathrm{span}\{v_{\mathrm{u}}\}$. Numerically,

$$C \simeq \begin{bmatrix} 0.99991 & 0.00884 & 0.01018 \\ -0.01026 & 0.01018 & 0.99990 \end{bmatrix}.$$

Thus, $\mathrm{rank}\, \mathcal{O}_2(C, A) = 2 < 3$ and $\kappa_C = 0$, so the full state is unobservable. Reported values are rounded, while computations use full precision.

We impose the two-sided sideslip constraint

$$|\beta_{\mathrm{s}}| \leq \beta_{\max}, \qquad \beta_{\max} = 0.25°.$$

Both constraints have relative degree two, with relevant unobservable polynomial $\mu(s) = s - \lambda_{\mathrm{u}}$.

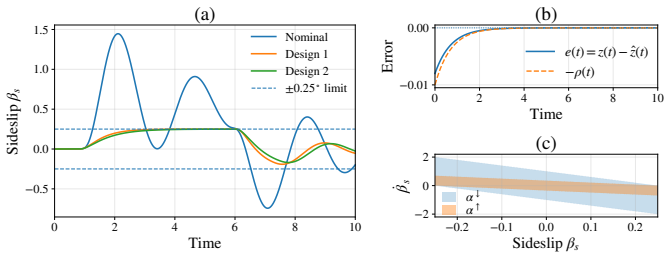


Fig. 1. Aircraft roll–yaw simulation: (a) sideslip response; (b) estimation error and bound; (c) HOCBF admissible regions for the two gain orderings.

For Design 1, choose $\phi^{(1)}(s) = (s+2)(s+4)$. Since $\phi^{(1)}(\lambda_u) \neq 0$, $\mu^{\mathrm{rem}}(s) = s - \lambda_u$. Hence, for these gains the HOCBF is detectable but not observable. Among observers satisfying (8), order one is necessary and sufficient, with unavoidable error pole λ_u and rate limit $\Gamma = -\lambda_u \approx 1.3902$. One realization for the upper constraint is

$$\dot{\eta} \simeq \lambda_u \eta + \begin{bmatrix} -0.0985 & 0.0041 \end{bmatrix} y - 0.1108u,$$
$$\hat{z} \simeq \eta + \begin{bmatrix} -0.9399 & 5.4810 \end{bmatrix} y.$$

For Design 2, choose $\phi^{(2)}(s) = (s-\lambda_u)(s+4)$. The hidden factor is now assigned to the HOCBF polynomial, so $\mu^{\mathrm{rem}} = 1$ and the HOCBF is observable. Hence, the sideslip constraint is HOCBF-observable. Since $\kappa_C = 0$, $\hat{z} \simeq \begin{bmatrix} 1.4209 & 4.8469 \end{bmatrix} y$, and no dynamic observer state is required. Thus, within the considered observer architecture, the required dynamic order drops from one to zero, illustrating Theorem 1.

For the simulations, $x(0) = 0$ and

$$u_{\mathrm{nom}}(t) = -0.11[\tanh(10(t-1)) - \tanh(10(t-6))] \text{ rad},$$

which violates the sideslip limit. Figure 1(a) shows a nominal peak of about $1.45°$, while both filters maintain $|\beta_s| \leq 0.25°$, confirming Proposition 2. For Design 1, $e(0) = -0.008$ and $\rho(t) = 0.01e^{\lambda_u t}$; the same ρ applies to both constraints. Figure 1(b) confirms the bound in Proposition 3 and the rate predicted by Theorem 2. The QP remains feasible although D is not full row rank.

Finally, permuting the gains $\{-\lambda_u, 4\}$ in Design 2 leaves the final HOCBF inequality and observer unchanged. The intermediate conditions are

$$-\alpha_1(\beta_{\max} + \beta_s) \leq \dot{\beta}_s \leq \alpha_1(\beta_{\max} - \beta_s).$$

The nondecreasing order has $\alpha_1 = -\lambda_u$, whereas the nonincreasing order has $\alpha_1 = 4$. Figure 1(c) shows the corresponding admissible regions in the $(\beta_s, \dot{\beta}_s)$ plane: the region for $\boldsymbol{\alpha}^{\downarrow}$ contains that for $\boldsymbol{\alpha}^{\uparrow}$, as predicted by Theorem 3.

VI. Conclusion

This letter developed an output-feedback HOCBF safety-filtering framework based on scalar functional observers, avoiding unnecessary full-state reconstruction. A polynomial co-design characterizes fixed-gain HOCBF observability and detectability, their achievability through gain selection, and observer-order and rate limits. Certified one-sided error bounds ensure safety, while gain ordering enlarges the HOCBF admissible set. Future work will consider nonlinear systems and shared functional observers.

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