Robinson's Counterexample and Regularity Properties of Optimization-Based Controllers

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Abstract

This paper studies technical properties of optimization-based controllers, which are obtained by solving optimization problems where the parameter is the system state and the optimization variable is the input to the system. We provide a collection of results about their regularity, as well as the existence and uniqueness of solutions of the closed-loop systems defined by them. In particular, we revisit Robinson's counterexample, which shows that, even for relatively well-behaved parametric optimization problems, the corresponding optimizer might not be locally Lipschitz with respect to the parameter. We show that controllers obtained from optimization problems whose objective and constraints have the same properties as those in Robinson's counterexample enjoy regularity properties that guarantee the existence (and in some cases, uniqueness) of solutions of the corresponding closed-loop system.

Keywords: Parametric optimization, optimization-based control, existence and uniqueness of solutions

1. Robinson's Counterexample

In [1], Robinson introduces the following parametric optimization problem: for $x = (x_1, x_2) \in \mathbb{R}^2$, consider

$$\min_{u\in\mathbb{R}^4} \frac{1}{2}u^\top u \tag{1a}$$

$$\text{s.t. } A(x)u \geq b(x) \tag{1b}$$

where

$$A(x) = \begin{bmatrix} 0 & -1 & 1 & 0, \\ 0 & 1 & 1 & 0, \\ -1 & 0 & 1 & 0, \\ 1 & 0 & 1 & x_1 \end{bmatrix}, \quad b(x) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1+x_2 \end{bmatrix}.$$

Problem (1) is a quadratic program with strongly convex objective function, smooth objective function and constraints, and for which Slater's condition [2, Section 5.2.3] holds for every value of the parameter (this can be shown by noting that $\hat{u} = (0, 0, 2+|x_2|, 0)$ satisfies all constraints strictly). Despite these nice properties, the parametric solution of (1) is not locally Lipschitz at $(x_1, x_2) = (0, 0)$. Indeed, let $u^* : \mathbb{R}^2 \to \mathbb{R}^4$ be the parametric solution of (1) and $u_4^* : \mathbb{R}^2 \to \mathbb{R}$ denote its fourth component, which is given by

$$u_4^*(x_1, x_2) = \begin{cases} 0 & \text{if } x_2 \leq 0, \\ \frac{x_2}{x_1} & \text{if } x_2 \geq 0,\ x_1 \neq 0, \frac{x_1^2}{2} \geq x_2, \\ \frac{x_1(x_2+1)}{x_1^2+2} & \text{otherwise.} \end{cases}$$

Figure 1 below depicts u_4^* numerically. The other components of u^* are continuously differentiable and therefore locally Lipschitz. However, if $p_{x_1} = (x_1, \frac{1}{2}x_1^2)$ and $q_{x_1} = (x_1, 0)$, we have

$$\frac{\|u_4^*(p_{x_1}) - u_4^*(q_{x_1})\|}{\|p_{x_1} - q_{x_1}\|} = \frac{1}{x_1}.$$

Since x_1 can be taken to be arbitrarily small, this shows that u^* is not locally Lipschitz at the origin.

2. Related Work: Parametric Optimization and Optimization-Based Controllers

This section reviews existing literature on parametric optimization and, specifically, its connec-

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tion to optimization-based controllers. The theory of parametric optimization [3, 4, 5] considers optimization problems that depend on a parameter and studies the regularity properties of the minimizers with respect to the parameter. Parametric optimization problems arise naturally in systems and control when designing optimization-based controllers, which are ubiquitous in numerous areas including safety-critical control [6], model predictive control [7, 8], and online feedback optimization [9, 10].

Given a system of the form:

$$\dot{x} = F(x, u) \tag{2}$$

with state $x \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$, an optimization-based controller is a feedback law obtained by solving a problem of the form

$$\underset{u \in \mathbb{R}^m}{\operatorname{argmin}} \; f(x, u) \tag{3a}$$

$$\text{s.t. } g(x, u) \leq 0 \tag{3b}$$

with $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$. Note that the system state x acts as a parameter in (3). Assuming that the optimizer of (3) is unique for every $x \in \mathbb{R}^n$, this defines a function $u^* : \mathbb{R}^n \to \mathbb{R}^m$, mapping each state to the optimizer of (3). The flexibility of this approach allows to encode desirable goals for controller synthesis both in the cost function f (e.g., minimum control effort) and in the constraints g (e.g., prescribed decrease of a control Lyapunov function [11] or forward invariance of a set through a control barrier function [6]). Once synthesized, the controller u^* can be used to close the loop on the system (2) (here, $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz). Given the definition (3), the theory of parametric optimization can be brought to bear on characterizing the regularity properties of u^*. These properties can then be used to certify existence and uniqueness of solutions of the closed-loop system

$$\dot{x} = F(x, u^*(x)). \tag{4}$$

For instance, if u^* is locally Lipschitz, then the right-hand side of (4) is locally Lipschitz too, and then the Picard-Lindelöf theorem [12, Theorem 2.2] guarantees existence and uniqueness of solutions for small enough times. It is in this context that Robinson's counterexample is problematic, because it shows that, even for optimization problems defined by well-behaved data (including the widespread quadratic programs employed in the design of safe [6] and stabilizing [11] controllers), the resulting controller might not be locally Lipschitz. Additionally, since uniqueness of solutions is a fundamental assumption in Nagumo's theorem [13] to establish forward invariance of a safe set $\mathcal{C}$ of interest, the use of non-Lipschitz controllers might result in closed-loop systems for which, even if the rest of the assumptions of Nagumo's theorem are met, some of the solutions might leave $\mathcal{C}$. All of this has motivated the study [14, 15, 16] of additional conditions (which we make precise later) on the data of the optimization problem that guarantee local Lipschitz continuity and even stronger regularity properties of optimization-based controllers. However, the results in [14] either require the rather strong assumption of *strict complementary slackness*, which is not satisfied in many cases of interest, or are limited to quadratic programs that satisfy a set of technical conditions. Finally, the regularity results in our previous work [15, 16] are limited to second-order convex programs.

3. Paper Contributions

This paper studies technical properties of optimization-based controllers. We provide a collection of results, most of them novel, and some from the literature, but restated here for completeness and from the perspective of optimization-based control. Our main motivation to write this paper was to provide an integrative presentation of insights and results about the regularity of optimization-based controllers.

On the technical level, the contributions of the paper are as follows. The first contribution seeks to characterize the regularity properties enjoyed by the parametric optimizer of problems defined by objective and constraints with the same properties as in Robinson's counterexample. This is important as confusion may arise in the literature due to the loose use of terminology. Indeed, according to [4, Theorem 6.4], a parametric optimization problem whose data satisfies the properties of Robinson's counterexample has a Lipschitz minimizer! This apparent contradiction is rooted in different notions of Lipschitz continuity, which this note clarifies precisely. We show that, under the conditions of Robinson's counterexample, even though the parametric optimizer is not necessarily locally Lipschitz, it enjoys other desirable regularity properties.

Our second contribution is to show that under these regularity properties, the existence (and in some cases, uniqueness) of solutions of the closed-loop system obtained with the corresponding optimization-based controller are guaranteed. We also introduce an example that shows that, in general, these conditions are not enough to guarantee uniqueness of solutions of the closed-loop system, and stronger conditions are required. For completeness, we place the results within the state of the art in the literature, illustrating how different properties on the optimization problem translate into regularity properties of the corresponding optimization-based controller and the existence and uniqueness of solutions of the closed-loop system. To the best of our knowledge, such a comprehensive collection of results is not available in the literature.

Finally, our last contribution is to show that under the conditions of Robinson's counterexample, the satisfaction of the sub-tangentiality condition in Nagumo's Theorem [13] for a set of interest does not guarantee that all solutions of the closed-loop system remain in the set. However, this can be guaranteed under slightly stronger assumptions, which we make precise.

4. Notions of Regularity of Functions

Throughout the note, we make use of the following notions of regularity of functions.

Definition 1. (Notions of Lipschitz continuity): *A function $f : \mathbb{R}^n \to \mathbb{R}^q$ is*

- point-Lipschitz at $x_0 \in \mathbb{R}^n$ *if there exists a neighborhood $\mathcal{U}$ of x_0 and a constant $L \geq 0$ such that*

$$\|f(x) - f(x_0)\| \leq L \|x - x_0\|, \quad \forall x \in \mathcal{U}. \quad (5)$$

- locally Lipschitz at $x_0 \in \mathbb{R}^n$ *if there exists a neighborhood $\tilde{\mathcal{U}}$ of x_0 and a constant $\tilde{L} \geq 0$ such that*

$$\|f(x) - f(y)\| \leq \tilde{L} \|x - y\|, \quad \forall x, y \in \tilde{\mathcal{U}}. \quad (6)$$

The notion of point-Lipschitz continuity is used, for instance, in [4, Section 6.3] and called *Lipschitz stability*, without clearly acknowledging the difference with the notion of local Lipschitz continuity. Studying point-Lipschitz continuity is natural in the context of parametric optimization, as one is normally interested in understanding the changes in the solution with respect to a *fixed* value of the parameter. Locally Lipschitz functions are point-Lipschitz, but the converse is not true. For instance, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x \sin(\frac{1}{x})$ if $x \neq 0$ and $f(0) = 0$ is point-Lipschitz but not locally Lipschitz at the origin.

Definition 2. (Hölder property): *A function $f : \mathbb{R}^n \to \mathbb{R}^q$ has the* Hölder property *at $x_0 \in \mathbb{R}^n$ if there exists a neighborhood $\hat{\mathcal{U}}$ of x_0 and constants $C > 0$, $\alpha \in (0, 1]$ such that*

$$\|f(x) - f(y)\| \leq C \|x - y\|^{\alpha}, \quad \forall x, y \in \hat{\mathcal{U}}. \quad (7)$$

Note that if f is locally Lipschitz at x_0 then it also has the Hölder property at x_0 but the converse is not true.

Definition 3. (Directionally differentiable function): *A function $f : \mathbb{R}^n \to \mathbb{R}$ is directionally differentiable if for any vector $v \in \mathbb{R}^n$, the limit*

$$\lim_{h \to 0} \frac{f(x + hv) - f(x)}{h}$$

exists. A vector-valued function is directionally differentiable if each of its components is directionally differentiable.

Let $\Omega \subset \mathbb{R}^n$. Throughout the paper, we say that a function $\varphi : \Omega \to \mathbb{R}^d$ belongs to the set $\mathcal{C}^k(\Omega)$ if φ is k-times continuously differentiable. In the case where Ω can be partitioned as $\Omega \subset \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$, and φ takes the form $(x, u) \mapsto \varphi(x, u)$, we say that $\varphi \in \mathcal{C}^{k,\ell}(\Omega)$ if φ is k-times continuously differentiable with respect to x and ℓ-times continuously differentiable with respect to u.

5. Regularity of Parametric Optimizers under the Properties of Robinson's Counterexample

We consider parametric optimization problems whose objective and constraints satisfy the same conditions as in Robinson's counterexample. The following result characterizes the regularity properties of the corresponding parametric optimizers.

Proposition 4. (Regularity Properties of Parametric Optimizer): *Suppose that f and g belong to $\mathcal{C}^{2,2}(\mathbb{R}^n \times \mathbb{R}^m)$. Further assume that given $x_0 \in \mathbb{R}^n$, $f(x_0, \cdot)$ is strongly convex, $g(x_0, \cdot)$ is convex and there exists $\hat{u} \in \mathbb{R}^m$ such that $g(x_0, \hat{u}) < 0$. Then,*

(i) *There exists a neighborhood* $\mathcal{V}_{x_0}$ *of* x_0 *such that* u^* *is point-Lipschitz at* y *for all* $y \in \tilde{\mathcal{V}}_{x_0}$*;*
(ii) u^* *has the Hölder property at* x_0*;*
(iii) u^* *is directionally differentiable at* x_0*.*

Proof. First we note that since $f(x_0, \cdot)$ is strongly convex and $g(x_0, \cdot)$ is convex for all x_0, $u^*(x_0)$ is unique and well-defined for all $x_0 \in \mathbb{R}^n$.

To prove (i) we use [4, Theorem 6.4]. The fact that there exists $\hat{u} \in \mathbb{R}^m$ such that $g(x_0, \hat{u}) < 0$ implies that Slater's Condition holds. Hence, by [17, Prop. 5.39], since $g(x_0, \cdot)$ is convex, the Mangasarian-Fromovitz Constraint Qualification (MFCQ) holds at $(x_0, u^*(x_0))$. Furthermore, since $f(x_0, \cdot)$ is strongly convex and $g(x_0, \cdot)$ is convex, the second-order condition SOC2 [4, Definition 6.1] holds. All of this, together with the twice continuous differentiability of f and g imply, by [4, Theorem 6.4], that u^* is point-Lipschitz at x_0. Now, since g is continuous, there exists a neighborhood $\mathcal{V}_{x_0}$ of x_0 such that $g(y, \hat{u}) < 0$ for all $y \in \mathcal{V}_{x_0}$. By repeating the same argument, u^* is point-Lipschitz at y for all $y \in \mathcal{V}_{x_0}$.

Now let us prove (ii). We use [18, Theorem 2.1], which gives a sufficient condition for the solution of a variational inequality to have the Hölder property. We first note that a constrained optimization problem of the form (3) can be posed as a variational inequality (cf. [19]). Since f is twice continuously differentiable and strongly convex, conditions (2.1) and (2.2) in [18, Theorem 2.1] hold. Moreover, since MFCQ holds at $(x_0, u^*(x_0))$ (because SC holds), by [20, Remark 3.6] the constraint set is pseudo-Lipschitzian [18, Definition 1.1]. All of this implies by [18, Theorem 2.1] that u^* has the Hölder property at x_0.

Finally, (iii) follows from the fact that SC implies MFCQ and [21, Theorem 1]. □

In Proposition 4, note that neither (i) implies (ii) nor the converse. Even though the parametric optimizer in Robinson's counterexample is not locally Lipschitz, Proposition 4 shows that it enjoys other, slightly weaker, regularity properties. In particular, this result implies that u_4^*, the fourth component of the parametric optimizer of Robinson's counterexample, is continuous, cf. Figure 1.

The following two examples show that the results from Proposition 4 do not hold if the assumptions are slightly weakened.

Example 5. (Discontinuous optimizer without Slater's condition): The following example, taken

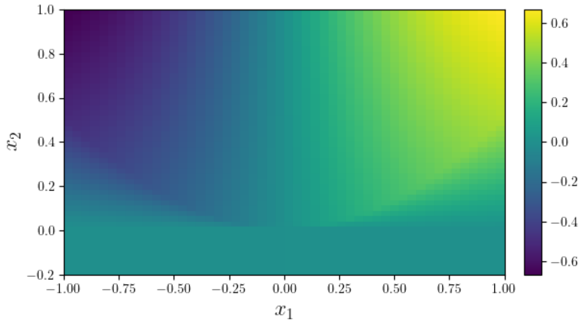

Figure 1: Numerical depiction of the fourth component of the parametric optimizer of Robinson's counterexample, cf. (1). The plot shows that it is continuous at the origin, in agreement with Proposition 4.

from [14, Section VI], shows that if Slater's condition does not hold, then continuity of the parametric optimizer is not guaranteed even if the rest of assumptions from Proposition 4 do hold:

$$u^*(x) = \underset{u \in \mathbb{R}}{\operatorname{argmin}} \frac{1}{2}u^2 - 2u, \tag{8a}$$

$$\text{s.t.} \quad xu \leq 0. \tag{8b}$$

Indeed, the objective function and constraint of (8) are twice continuously differentiable, the objective function is strongly convex and the constraint is convex for any $x \in \mathbb{R}$. However, Slater's condition does not hold at $x = 0$. In fact, $\hat{u}^*$, which is given by

$$\hat{u}^*(x) = \begin{cases} 2 & \text{if} \quad x \leq 0, \\ 0 & \text{else}, \end{cases}$$

is discontinuous at $x = 0$. •

Example 6. (Not point-Lipschitz optimizer without differentiability of problem data with respect to the parameter): If f and g are not differentiable with respect to the parameter x but the rest of the assumptions of Proposition 4 hold, the following example, inspired by Robinson's counterexample, shows that the parametric optimizer is not necessarily point-Lipschitz. Let $x = (x_1, x_2) \in \mathbb{R}^2$ and consider (1) with

$$A(x) = \begin{bmatrix} 0 & -1 & 1 & 0, \\ 0 & 1 & 1 & 0, \\ -1 & 0 & 1 & 0, \\ 1 & 0 & 1 & \sqrt{|x_1|} \end{bmatrix}, \quad b(x) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 + x_2 \end{bmatrix}.$$

Let $\tilde{u}^* : \mathbb{R}^2 \to \mathbb{R}^4$ be its parametric solution and let $\tilde{u}_4^* : \mathbb{R}^2 \to \mathbb{R}$ denote its fourth component, which is given by

$$\tilde{u}_4^*(x) = \begin{cases} 0 & \text{if } x_2 \leq 0, \\ \frac{x_2}{\sqrt{|x_1|}} & \text{if } x_2 \geq 0,\ x_1 \neq 0, \frac{|x_1|}{2} \geq x_2, \\ \frac{\sqrt{|x_1|}(x_2+1)}{|x_1|+2} & \text{otherwise.} \end{cases}$$

Let $x_1 > 0$ and define $p_{x_1} = (x_1, \frac{|x_1|}{2})$. Note that

$$\frac{\|\tilde{u}_4^*(p_{x_1}) - \tilde{u}_4^*(0)\|}{\|p_{x_1} - 0\|} = \frac{1}{\sqrt{5|x_1|}}.$$

Since x_1 can be taken to be arbitrarily small, $\tilde{u}^*$ is not point-Lipschitz at the origin. However, because f and g, as well as their first and second derivatives with respect to u, are continuous in u and x, and the rest of assumptions of Proposition 4 hold, then by [22, Theorem 5.3], the corresponding parametric optimizer, $\tilde{u}_4^*$, is continuous. •

6. Existence and Uniqueness of Solutions under Optimization-Based Controllers

Here, we leverage the regularity properties established in Section 5 to study existence and uniqueness of solutions for the closed-loop system under the optimization-based controller. The following result establishes existence of solutions.

Proposition 7. (Existence of solutions for the closed-loop system): *Suppose that f and g are twice continuously differentiable in $\mathbb{R}^n \times \mathbb{R}^m$. Further assume that given $x_0 \in \mathbb{R}^n$, $f(\cdot, x_0)$ is strongly convex, $g(\cdot, x_0)$ is convex and there exists $\hat{u} \in \mathbb{R}^m$ such that $g(\hat{u}, x_0) < 0$. Let $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be locally Lipschitz. Then, there exists $\delta > 0$ such that the differential equation (4) has at least one solution $x : (-\delta, \delta) \to \mathbb{R}^n$ with initial condition $x(0) = x_0$.*

Proof. By Proposition 4, u^* has the Hölder property at x_0 and there exists a neighborhood $\mathcal{V}_{x_0}$ of x_0 such that u^* is point-Lipschitz at y for all $y \in \mathcal{V}_{x_0}$. Both of these properties imply that u^* is continuous in a neighborhood of x_0. The result follows by Peano's existence theorem [23, Theorem 2.1]. □

Next, we study uniqueness of solutions. The question we address is whether the assumptions of Proposition 7 are sufficient to ensure this property. We first note that the Hölder property does not imply uniqueness, even in simple one-dimensional examples. For example, the differential equation $\dot{x} = x^{1/3}$ has the Hölder property at 0 but infinitely many solutions starting from the origin. The next example shows that, in general, point-Lipschitz continuity does not imply uniqueness of solutions either.

Example 8. (Point-Lipschitz differential equation with non-unique solutions): Let $u^* : \mathbb{R}^2 \to \mathbb{R}^4$ be the parametric optimizer of Robinson's counterexample. Consider the dynamical system

$$\dot{x}_1 = \frac{1}{2}, \tag{9a}$$

$$\dot{x}_2 = u_4^*(x_1, x_2), \tag{9b}$$

with initial condition $(x_1(0), x_2(0)) = (0, 0)$. By Proposition 4, the vector field in (9) is point-Lipschitz at the origin. However, (9) admits the following two distinct solutions starting from the origin: $y_1(t) := (\frac{1}{2}t, 0)$ and $y_2(t) := (\frac{1}{2}t, \frac{1}{8}t^2)$, cf. Figure 2. •

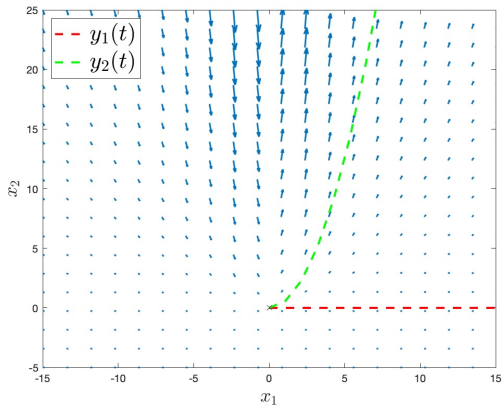

Figure 2: The blue arrows depict the vector field (9). The dashed red and green curves depict the two solutions y_1 and y_2 starting from the origin, where the vector field is point-Lipschitz but not locally Lipschitz.

Interestingly, the next result shows that point-Lipschitz continuity guarantees uniqueness of solutions starting from equilibria.

Proposition 9. (Point-Lipschitz continuity and Uniqueness): *Let $\tilde{F} : \mathbb{R}^n \to \mathbb{R}^n$ be point-Lipschitz at $x_0 \in \mathbb{R}^n$ and $\tilde{F}(x_0) = 0$. Then the function $x(t) = x_0$ for all $t \geq 0$ is the unique solution to the differential equation $\dot{x} = \tilde{F}(x)$ with initial condition $x(0) = x_0$.*

Proof. Let $\delta > 0$ and L be the point-Lipschitz continuity constant of $\tilde{F}$ and take $\delta < \frac{1}{L}$. Suppose that there exists another solution $y : [0, \delta) \to \mathbb{R}^n$ starting from x_0. Then, $\sup_{t\in[0,\delta)} \|y(t) - x_0\| > 0$. Moreover,

$$\sup_{t\in[0,\delta)} \|y(t) - x_0\| = \sup_{t\in[0,\delta)} \left\| \int_0^t \tilde{F}(y(s))ds \right\| =$$

$$\sup_{t\in[0,\delta)} \left\| \int_0^t \Big(\tilde{F}(y(s)) - \tilde{F}(x_0) \Big) ds \right\| \leq$$

$$\sup_{t\in[0,\delta)} \int_0^t L \|y(s) - x_0\| \, ds \leq$$

$$L\delta \sup_{t\in[0,\delta]} \|y(t) - x_0\| < \sup_{t\in[0,\delta]} \|y(t) - x_0\|$$

where in the last inequality we have used the fact that $\sup_{t\in[0,\delta)} \|y(t) - x_0\| > 0$. We hence reach a contradiction, which means that the constant solution is the only solution for $t \in [0, \delta)$. By repeating the same argument at time δ, we can extend this constant solution for all positive times. □

This result implies that in one dimension point-Lipschitz ODEs have unique solutions.

Corollary 10. (Point-Lipschitz continuity implies uniqueness in one dimension): *Let $\tilde{F} : \mathbb{R} \to \mathbb{R}$ be continuous in a neighborhood of x_0 and point-Lipschitz at x_0. Then, the differential equation $\dot{x} = \tilde{F}(x)$ with initial condition $x(0) = x_0$ has a unique solution.*

Proof. If $\tilde{F}(x_0) \neq 0$, by [24, Theorem 1.2.7], the differential equation has only one solution. If $\tilde{F}(x_0) = 0$, the result follows from Proposition 9. □

Since in general the assumptions of Proposition 7 are not sufficient to ensure uniqueness of solutions of the closed-loop system, additional assumptions must be made. Strengthening these assumptions has been explored in the literature [14] of optimization-based controllers. Under the assumption that both MFCQ and the *constant-rank constraint qualification* (CRCQ) hold, the parametric solution u^* is locally Lipschitz [25, Theorem 3.6] and the closed-loop system has a unique solution. The same conclusion can be obtained under the *linear independence constraint qualification* (LICQ) [26, Theorem 4.1]. Moreover, under the additional assumption of *strict complementary slackness* (SCS) [27, Theorem 2.1], the parametric solution u^* is continuously differentiable, so the closed-loop system has unique solutions. This last point was already noted in [14, Theorem 1].

On the other hand, if Slater's condition does not hold but the rest of assumptions of Proposition 7 hold, Example 5 shows that u^* can be discontinuous, in which case neither existence nor uniqueness of solutions is guaranteed. In the case where f and g are not differentiable with respect to the parameter, but the rest of assumptions of Proposition 7 hold, Example 6 shows that u^* is continuous but not necessarily point-Lipschitz. Therefore, in this case existence is guaranteed but uniqueness is not. Finally, if f is not strongly convex or g is not convex, the optimizer u^* is not guaranteed to be single-valued, which means that the usual notions of regularity of the controller and of solutions of the closed-loop system are not well defined. Table 1 summarizes these results.

7. Forward Invariance Properties of Optimization-Based Controllers

In this section we study conditions that guarantee the forward invariance of a set for the closed-loop system under an optimization-based controller. First, let us recall the notion of *tangent cone* to a set $\mathcal{C} \subset \mathbb{R}^n$:

Definition 11. *The tangent cone to $\mathcal{C} \subset \mathbb{R}^n$ at $x \in \mathbb{R}^n$ is*

$$T_{\mathcal{C}}(x) = \left\{ v \in \mathbb{R}^n \;\middle|\; \liminf_{h\to 0} \frac{\operatorname{dist}(x + hv, \mathcal{C})}{h} = 0 \right\}.$$

The basic result concerning forward invariance is the following:

Theorem 12. (Nagumo's Theorem [13, 28]): *Let $\tilde{F} : \mathbb{R}^n \to \mathbb{R}^n$, and consider the system $\dot{x} = \tilde{F}(x)$, and assume that, for each initial condition in a set $\mathcal{D} \subset \mathbb{R}^n$, it admits a globally unique solution. Let $\mathcal{C} \subset \mathbb{R}^n$ be a closed set. Then the set $\mathcal{C}$ is forward invariant for the system if and only if $\tilde{F}(x) \in T_{\mathcal{C}}(x)$ for all $x \in \mathcal{C}$.*

The condition that $\tilde{F}(x) \in T_{\mathcal{C}}(x)$ for all $x \in \mathcal{C}$ is called the *sub-tangentiality condition*, and can be enforced using the constraints of an optimization-based feedback controller of the form (3). Indeed, suppose that $\mathcal{C}$ is parameterized as $\mathcal{C} = \{x \in \mathbb{R}^n \mid h_j(x) \geq 0, \; 1 \leq j \leq p\}$, where $h_j : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable for $j = 1, \ldots, p$, and the dynamics take the form

$$\dot{x} = F(x, u) = F_0(x) + \sum_{i=1}^{m} u_i F_i(x) \qquad (10)$$

Assumptions	Single-valued	Regularity of u^*	Existence	Uniqueness
$f, g \in \mathcal{C}^{2,2}(\mathbb{R}^n \times \mathbb{R}^m)$ $f(x_0, \cdot)$ strongly convex $g(x_0, \cdot)$ convex LICQ and SCS	✓	$\mathcal{C}^1(\mathbb{R}^m)$ cf. [27]	✓	✓
$f, g \in \mathcal{C}^{2,2}(\mathbb{R}^n \times \mathbb{R}^m)$ $f(x_0, \cdot)$ strongly convex $g(x_0, \cdot)$ convex LICQ	✓	Locally Lipschitz cf. [26]	✓	✓
$f, g \in \mathcal{C}^{1,2}(\mathbb{R}^n \times \mathbb{R}^m)$ $f(x_0, \cdot)$ strongly convex $g(x_0, \cdot)$ convex CRCQ and MFCQ	✓	Locally Lipschitz cf. [25]	✓	✓
$f, g \in \mathcal{C}^{1,2}(\mathbb{R}^n \times \mathbb{R}^m)$ $f(x_0, \cdot)$ strongly convex $g(x_0, \cdot)$ convex Slater's condition	✓	cf. Proposition 4	✓	Only in special cases cf. Proposition 9 Corollary 10, Example 9
$f, g \in \mathcal{C}^{0,2}(\mathbb{R}^n \times \mathbb{R}^m)$ $f(x_0, \cdot)$ strongly convex $g(x_0, \cdot)$ convex Slater's Condition	✓	Continuous, but might not be point-Lipschitz cf. Example 6	✓	✗
$f, g \in \mathcal{C}^{2,2}(\mathbb{R}^n \times \mathbb{R}^m)$ $f(x_0, \cdot)$ strongly convex $g(x_0, \cdot)$ convex	✓	Might be discontinuous cf. Example 5	✗	✗
$f, g \in \mathcal{C}^{2,2}(\mathbb{R}^n \times \mathbb{R}^m)$	✗	—	—	—

Table 1: Table showcasing the properties of optimization-based controllers under different assumptions. The first column describes the different assumptions. The second column describes whether the optimizer u^* is guaranteed to be single-valued. The third column describes the regularity properties of u^*. The fourth (resp. fifth) column describes whether existence (resp. uniqueness) of classical solutions of the closed-loop system (4) is guaranteed (provided that $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz). Recall that LICQ stands for *linear independence constraint qualification*, SCS stands for *strict complementary slackness*, MFCQ stands for *Mangasarian-Fromovitz Constraint Qualification* and CRCQ stands for *constant rank constraint qualification*.

for smooth functions $F_i : \mathbb{R}^n \to \mathbb{R}^n$ for $i \in \{0, \dots, m\}$. Next, define $A(x) \in \mathbb{R}^{p \times m}$ and $b(x) \in \mathbb{R}^p$ as

$$A(x) = \begin{bmatrix} \mathcal{L}_{F_1} h_1(x) & \dots & \mathcal{L}_{F_m} h_1(x) \\ \vdots & \ddots & \vdots \\ \mathcal{L}_{F_1} h_p(x) & \dots & \mathcal{L}_{F_m} h_p(x) \end{bmatrix}$$

$$b(x) = \begin{bmatrix} -\alpha(h_1(x)) - \mathcal{L}_{F_0} h_1(x) \\ \vdots \\ -\alpha(h_m(x)) - \mathcal{L}_{F_0} h_m(x), \end{bmatrix}$$

where α is a class-$\mathcal{K}$ function. Let $A_j(x)$ denote the jth row of $A(x)$, and for $J \subset \{1, \dots, p\}$, let $A_J(x)$ denote the matrix consisting of the rows of $A(x)$ corresponding to $j \in J$.

In the literature on optimization-based control design [29], the feasibility of the system $A_j(x)u \geq b_j(x)$ for all $x \in \mathbb{R}^n$ such that $h_j(x) \geq 0$ is equivalent to h_j being a *control barrier function* for the set $\{x \in \mathbb{R}^n \mid h_j(x) \geq 0\}$. Since we are considering the case where $\mathcal{C}$ is possibly parameterized by multiple inequalities, here we make the stronger assumption that the system $A(x)u \geq b(x)$ (where the inequality holds componentwise) is feasible for all $x \in \mathcal{C}$. In this case, if $\mathcal{C}$ satisfies an appropriate constraint qualification condition (e.g., MFCQ or LICQ) and $u^* : \mathbb{R}^n \to \mathbb{R}^m$ is a feedback controller such that $A(x)u^*(x) \geq b(x)$ for all $x \in \mathcal{C}$, then the closed-loop dynamics satisfies the subtangentiality condition $F(x, u^*(x)) \in T_{\mathcal{C}}(x)$. Such a controller can be obtained from the solution of a parametric optimization problem of the form (3) where $g(x, u) = b(x) - A(x)u$.

To show invariance using Theorem 12, one needs to additionally ensure that the closed-loop dynamics has unique solutions. The conditions discussed in Section 6 and summarized in Table 1 can be translated into easily checkable conditions on the objective function and $A(x)$ and $b(x)$. The following result uses [25, Theorem 3.6] to ensure uniqueness, and therefore forward invariance.

Theorem 13. (Sufficient conditions for forward invariance of $\mathcal{C}$ with respect to closed-loop dynamics): *Consider the dynamics* (10)*, and the optimization problem* (3) *where* $f \in \mathcal{C}^{1,2}(\mathbb{R}^n \times \mathbb{R}^m)$ *is strongly convex, and* $g(x,u) = b(x) - A(x)u$*. If*

- *For all* $x \in \mathcal{C}$*, there exists* $u \in \mathbb{R}^m$ *such that* $A(x)u > b(x)$*, and*
- *For all* $x \in \mathcal{C}$*, there is an open set* $U_x \subset \mathbb{R}^n$ *containing* x *such that, for all* $J \subset \{1,\dots,p\}$*, the matrix* $A_J(y)$ *has constant rank for all* $y \in U_x$*,*

Then the closed-loop system under the feedback controller specified by the solution to (3) *has unique solutions, and* $\mathcal{C}$ *is forward invariant.*

In the case where the closed-loop dynamics are point-Lipschitz, solutions are not necessarily unique and therefore forward invariance of $\mathcal{C}$ cannot be guaranteed by Theorem 12. In fact, the following is an example of a system where the sub-tangentiality condition holds but there exist solutions starting in $\mathcal{C}$ that eventually leave.

Example 14. (Point-Lipschitz differential equation violating forward invariance): Let $\mathcal{C} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \leq 0\}$ and consider the system with the feedback controller defined in Example 9. Because $\mathcal{C}$ satisfies LICQ, the tangent cone can be computed as $T_{\mathcal{C}}(x_1,x_2) = \mathbb{R}^2$ if $x_2 < 0$, and $T_{\mathcal{C}}(x_1, 0) = \{(\xi_1,\xi_2) \mid \xi_2 \leq 0\}$. The closed-loop system satisfies $F(x, u^*(x)) = (\frac{1}{2}, u_4^*(x_1,x_2)) \in T_{\mathcal{C}}(x_1,x_2)$ for all $(x_1,x_2) \in \mathcal{C}$. However, the solution $y_2(t) = (\frac{1}{2}t, \frac{1}{8}t^2)$ satisfies $y_2(0) \in \mathcal{C}$ and $y_2(t) \notin \mathcal{C}$ for all $t > 0$. •

Example 9 is problematic because it shows that even if the *sub-tangentiality* condition for a safe set $\mathcal{C}$ is included as one of the constraints of the optimization-based controller, if the solutions of the closed-loop system are not unique, some of the solutions might leave the safe set $\mathcal{C}$. However, using the notion of minimal barrier functions [30], the following result gives a condition for forward invariance that can be applied to systems with non-unique solutions.

Theorem 15. (Minimal Barrier Functions, [30, Theorem 1]): *Let* $\tilde{F} : \mathbb{R}^n \to \mathbb{R}^n$ *be a continuous function and consider the system* $\dot{x} = \tilde{F}(x)$*. Let* $h : \mathbb{R}^n \to \mathbb{R}$ *be a continuously differentiable function and let* $\mathcal{C} = \{x \in \mathbb{R}^n \mid h(x) \geq 0\}$ *be a nonempty set. If* h *is a minimal barrier function (cf. [30, Definition 2]), then any solution of* $\dot{x} = \tilde{F}(x)$ *with initial condition in* $\mathcal{C}$ *remains in* $\mathcal{C}$ *for all positive times.*

A simple scenario in which h is a minimal barrier function is if there exists an strictly increasing function $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$ and an open set $\mathcal{D}$ with $\mathcal{C} \subset \mathcal{D}$ such that $\nabla h(x)^\top \tilde{F}(x) \geq -\alpha(h(x))$ for all $x \in \mathcal{D}$. Such a set $\mathcal{D}$ and class $\mathcal{K}$ function α cannot be found in Example 14. Since Theorem 15 only requires $\tilde{F}$ to be continuous, the system $\dot{x} = \tilde{F}(x)$ might have multiple solutions starting from the same initial condition. However, the result ensures that if the initial condition is in $\mathcal{C}$, then all solutions remain in $\mathcal{C}$ for all positive times. Moreover, since point-Lipschitz functions are continuous, Theorem 15 can be applied to differential equations defined by point-Lipschitz functions. Therefore, if one of the constraints in (3) corresponds to the minimal control barrier function condition (cf. [30, Definition 3]) of a function h, and if the resulting controller is point-Lipschitz (e.g., by satisfying the hypothesis of Proposition 4), then all solutions of the closed-loop system that start in $\mathcal{C} := \{x \in \mathbb{R}^n \mid h(x) \geq 0\}$ remain in $\mathcal{C}$ for all positive times.

8. Conclusions

This note has studied different technical properties of optimization-based controllers. We have provided an integrative presentation of insights and results about the regularity properties of such controllers, as well as the existence and uniqueness of solutions of closed-loop systems defined by them. In particular, we have sought to clarify the properties enjoyed by parametric optimizers arising from optimization problems whose data satisfies the same properties as in Robinson's counterexample. We have shown that, even though the parametric optimizer in Robinson's counterexample is not locally Lipschitz, it enjoys other important regularity properties, like point-Lipschitz continuity. These are enough to guarantee existence of solutions of dynamical systems driven by optimization-based controllers but, in general, not uniqueness (for which otherwise stronger constraint qualifications must be satisfied), as we have illustrated with an example.

We have identified cases where point-Lipschitz continuity is enough to guarantee uniqueness of solutions. Finally, we have also studied conditions that ensure that all (not necessarily unique) solutions of the closed-loop system obtained from an optimization-based controller remain in a safe set of interest. The results presented in this note open the possibility of studying weaker conditions on the optimization problem that guarantee existence of solutions of the closed-loop system, as well as forward invariance guarantees for those solutions, possibly also using notions of solutions for discontinuous systems like Carathéodory or Krasovskii solutions.

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